

# Thomae formula for Abelian covers of $\mathbb{CP}^1$

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## Abstract

Let  $X$  be an Abelian cover of  $\mathbb{CP}^1$ , with Galois group  $A$ . We show how the results of [KZ], producing a class of non-positive divisors on  $X$  of degree  $g - 1$  supported on the pre-images of the branch points on  $X$  such that the Riemann theta function does not vanish on their image in  $J(X)$ , allows one to state and prove a Thomae type formula similar to the formulas from [BR1], [Na], [Z], [EG2] and [Ko2]. This formula links polynomial function on branch points with the value of theta function with the divisor we defined above, such that the resulting matrix, up to a certain determinant, is constant on the moduli space of such covers, with given discrete parameters.

## Introduction

The references [Th1] and [Th2] from the 19th century found a relation between the even theta constants on a hyper-elliptic Riemann surface  $X$  and certain polynomials in the branching values. More explicitly, for every even theta characteristics  $e$  there exists a polynomial  $p_e$  in the branching values of  $X$  such that the quotient  $\theta^8[e](0, \tau)/p_e$  is independent of the choice of  $e$ . Moreover, this common quotient was shown in these references to be invariant under perturbations of the branching values, when dividing by a certain determinant. These formulae are now known, after the author of these references, as *Thomae formulae*. They can be interpreted as the quotient being a constant parameter on the moduli space of hyper-elliptic curves of a given genus.

After laying dormant for about a century, the Thomae formulae returned to active research, partly due to emerging interest from physics, and generalizations of them in several directions have been established. The first natural generalization of the hyper-elliptic equation is the non-singular  $Z_n$  equation, in which one adds the  $n$ th root of a polynomial with  $nm$  distinct roots. The corresponding Thomae formulae, using rational theta characteristics, was established in [BR1], and then more rigorously in [Na]. The independence of the characteristics was established via elementary means in [EiF], and then extended to the general non-singular case in [EbF] (we also mention the contributions of [M] and [MT] to the case  $n = 3$ , as well as the application of these formulae with  $n = 3$  to Young tableaux). After the construction of Thomae type formulae

for a family of singular  $Z_n$  curves in [EG1] (based on their earlier work [EG2]), the book [FZ] has shown how to obtain the independence of the characteristics using the elementary tools for showing the invariance of the characteristic in all the cases considered before, as well as for some other families of  $Z_n$  curves.

The case of general fully ramified  $Z_n$  curves was completed by the authors of the current paper, see [Ko2] for the invariance under perturbations and [Z] for the independence of the characteristics. Then arose the question: To which more general types can one state and prove Thomae type formulae? The independence of the characteristic seems to be more difficult (since full ramification in  $Z_n$  curves allows one to have invariant points on  $X$ ), but as for constructing a quotient that will be invariant under perturbations, this goal was achieved by the first author in [Ko3], for covers that are no longer cyclic but Abelian of exponent 2. The tools remain the ones from [Na].

In the previous paper [KZ] we have established a formalism for working with general Abelian covers of  $\mathbb{CP}^1$  (as part of a more general theory, with several applications), in which both the Abelian group  $A$  of deck transformation and its dual group  $\hat{A}$  play a role. We have also determined the set of non-special  $A$ -invariant divisors (representing theta characteristics) in terms of certain cardinality conditions (note, however, that the existence of divisors satisfying these conditions is not considered—see [GDT] for examples of  $Z_n$  curves, with prime  $n$ , for which no such divisors exist). This involves a different presentation of the branching values: Apart from belonging to an exponent in some polynomial (which depends on the choice of the polynomial), to every branching value we attach a non-trivial element of  $A$  that generates its stabilizer (but the choice of generator is, as is evident already in the  $Z_n$  curve case, important), an set of invariants of the cover  $X$  that is a refinement of the well-known signature.

The general search for such formulas is interesting at its own right, but it also has deep connections to other problems in mathematics, for which it may have applications of the sorts that we list below. First, the original Thomae formula was used in [dJ] in obtaining explicit forms of the Mumford isomorphism between the Hodge bundle to a certain power and the Tangent bundle of the moduli space of curves in the hyper-elliptic case. This result indicates a deep connection to arithmetic geometry, as it enables us to obtain expressions for Falting's invariant  $\delta$  (see, e.g., [W] for the evaluation of the latter invariant). Obtaining a more general Thomae formulae may therefore be the first step for generalizing these invariants further. As another application of the Thomae formulae, now for defining analogues of discriminants of hyper-elliptic curves in order to apply Mestre's AGM algorithm for counting points over finite fields, see [LL]. The initial step of this fast algorithm, which involves the duplication formula of theta functions, was the calculation of theta values using Thomae's formula for hyper-elliptic curves. Here an algebraic transformation should replace the duplication formula for theta functions, possibly allowing the extension of Mestre's algorithm for counting points on the more general curves considered here. In another direction, Thomae type formulas may be of interest over other number fields: See, e.g., the argument used by [Te] over the  $p$ -adic numbers

for computing the  $p$ -adic periods of modular curves. The generalization of the formalism developed in the current paper to  $p$ -adic fields, thus allowing the extension of the results of [Te] to other cases, is challenging, but in our opinion doable. In physics, Thomae type formulae should be instrumental solving the Riemann–Hilbert problem for a wider class of curves (see [EG1] for this type of application).

The main goal of this paper is to establish, on any Abelian cover  $X$  of  $\mathbb{CP}^1$ , the Thomae quotients attached to the appropriate rational characteristics on  $X$  that are invariant under perturbations of the branching values, i.e., which are constants on the moduli space for Abelian covers  $X$  with given invariants (i.e., refined signature). The method is the same as the one used in [Na], [Ko2], [Ko3], and others, but as the setting here is more general, we explain the method in more detail to clarify that it does work in this more general setting. The idea is to obtain explicit algebraic expressions for analytical objects such as the Szegő kernel function (with our specific characteristics) and the canonical differential on  $X \times X$ , and use a general relation from [Fa] to deduce equalities involving derivatives of theta functions. Using the Rauch Variational Formula from [Ra], this produces a differential equation for the theta constant (as a function of the branching values), from which the final result is then established. The Thomae formula now reads (see Theorem 6.6 for the precise statement)

$$\theta[e]^{8m}(0, \tau) = \alpha_e (\det C)^{4m} \prod_{(\sigma, j) < (\rho, i)} (\lambda_{\sigma, j} - \lambda_{\rho, i})^{\frac{4mn}{o(\sigma)o(\rho)} [2\phi_{h+d\mathbb{Z}}(s) + \phi_{h+d\mathbb{Z}}(0) + c_{\sigma, \rho}]}$$

where  $n = |A|$ ,  $m$  is the exponent of  $A$ ,  $o(\sigma)$  and  $o(\rho)$  are the orders of the non-trivial elements  $\sigma$  and  $\rho$  of  $A$  respectively,  $c_{\sigma, \rho}$  is a shorthand for  $\frac{(o(\sigma)-1)(o(\rho)-1)}{4}$ , and the two expressions involving  $\phi$  are certain generalized Dedekind sums arising from  $\sigma$  and  $\rho$ . The occurrence of these generalized Dedekind sums may suggest that the resulting polynomials may be connected to other interesting objects. We also prove an initial step in the direction of independence of the characteristics, as well as the independence of this quotient with respect to altering the map  $z$  by a Möbius transformation, two steps that ensure that these constants  $\alpha_e$  are defined on the moduli space of  $A$ -coverings of  $\mathbb{CP}^1$  with fixed discrete parameters. The full independence of this constant of  $e$  is left for future research.

This paper is divided into 7 sections. Section 1 summarizes some needed information from [KZ], including the form of the invariant divisors defining our characteristics. In Section 2 we give some results about the theta constants appearing in our analysis. Section 3 constructs the Szegő kernel using algebraic parameters, and evaluates its expansion around the diagonal. Section 4 investigates the form of the canonical differential in our case, including the Bergman projective kernel arising from its expansion around the diagonal. In Section 5 we consider the expansions around branch points, which using the formula from [Fa] gives the initial relation. Section 6 then relates the terms appearing in the expansions from the previous sections to derivatives of objects with respect to perturbing a branching value, and proves the main result. Finally, in Section 7

we establish a partial independence of the characteristic and show that the main theorem can be interpreted as well-definedness on the required moduli space, as well as formulate the conjecture about the independence of the characteristics in general.

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## 1 Non Special Divisors on Abelian Covers

Let  $z : X \mapsto \mathbb{CP}^1$  be an Abelian cover of the Riemann sphere. This means that there is a finite Abelian group  $A$  of automorphisms of  $X$ , whose order we denote by  $n$ , such that a point  $P \in X$  maps via  $z$  to the same value in  $\mathbb{CP}^1$  as any of its images under  $A$ , and that any automorphism of  $X$  which commutes with  $z$  lies in  $A$ . In the paper [KZ] (see Lemma 1.4 there) we showed that if  $A$  decomposes as the direct product of  $q$  cyclic groups  $H_l$ ,  $1 \leq l \leq q$  and  $|H_l| = m_l$  (so that  $\prod_{l=1}^q m_l = n$ ) then  $X$  is a fiber product of  $q$  cyclic covers of  $\mathbb{CP}^1$ , where the  $l$ th “component” of the fiber product, which we denote by  $Y_l$ , is a  $Z_{m_l}$ -curve. In order to give explicit expressions below we shall follow the general, abstract argument with an example that is based on such a decomposition. The equation of the  $Z_{m_l}$ -curve  $Y_l$  is of the form  $y_l^{m_l} = F_l(z)$ , where  $F_l(z)$  is a rational function on  $\mathbb{CP}^1$ , which is not a  $d$ th power in  $\mathbb{C}(z)$  for any divisor  $d$  of  $m_l$  (for the irreducibility of  $Y_l$ ). The irreducibility of the fibered product  $X$  is equivalent to the condition that for any sequence of integers  $\{e_l\}_{l=1}^q$  with  $c = \text{lcm}\{\frac{m_l}{\gcd\{m_l, e_l\}} \mid 1 \leq l \leq q\}$  and  $c > 1$  (so that not every  $e_l$  is a multiple of the corresponding  $m_l$ ) then  $\prod_{l=1}^q F_l^{e_l c / m_l}$  is not a  $c$ th power in  $\mathbb{C}(z)$ . We shall also make throughout the paper the simplifying assumption that  $\infty$  is not a ramification point of any of  $Y_l$ , hence also not of  $X$ . This condition is equivalent to  $m_l \mid \deg F_l$  for every  $l$ , where by the degree of rational function here we mean the degree of the numerator minus that of the denominator.

More abstractly, Section 1 of [KZ] divides the branching values of any Abelian cover  $z : X \rightarrow \mathbb{CP}^1$  into sets corresponding to non-trivial elements  $\sigma \in A$ , where for every such  $\sigma$  (whose order in  $A$  we denote by  $o(\sigma)$ ) there are  $r_\sigma$  points associated with  $\sigma$ , denoted  $\lambda_{\sigma,j}$  with  $1 \leq j \leq r_\sigma$ , such that in particular the stabilizer of any pre-image of  $\lambda_{\sigma,j}$  is generated by  $\sigma$  (the stabilizer of any point on  $X$  not mapping to any  $\lambda_{\sigma,j}$  is trivial). We shall use  $z^{-1}(\lambda_{\sigma,j})$  for the divisor, of degree  $\frac{n}{o(\sigma)}$ , consisting of those points on  $X$  mapping to  $\lambda_{\sigma,j}$  (with no multiplicity), as well as allow ourselves the slight abuse of notation by using the same (more broadly known) notation for the set containing these points. The same applies for the set (and divisor)  $z^{-1}(\infty)$  of poles of  $z$ , which contain  $n$  distinct points by our assumption of no branching over  $\infty$ . The Riemann–Hurwitz formula was seen in [KZ] to produce the following result.

**Proposition 1.1.** *The genus  $g$  of  $X$  equals  $1 - n + \sum_\sigma \frac{n r_\sigma}{o(\sigma)} (o(\sigma) - 1)$ .*

The character group (or dual group)  $\text{Hom}(A, \mathbb{C}^\times)$  of  $A$  is denoted by  $\hat{A}$ , and

satisfies  $|\widehat{A}| = n$  as well. For every character  $\chi \in \widehat{A}$  and  $\sigma \in A$  we associate the number  $0 \leq u_{\chi,\sigma} < o(\sigma)$  such that  $\chi(\sigma) = \mathbf{e}\left(\frac{u_{\chi,\sigma}}{o(\sigma)}\right)$ , where  $\mathbf{e}(w)$  stands for  $e^{2\pi i w}$  for any complex number  $w$ . [KZ] shows (the details are given in Sections 2 and 5 there) that the number  $t_\chi = \sum_\sigma \frac{r_\sigma u_{\chi,\sigma}}{ord(\sigma)}$  is a non-negative integer for every  $\chi \in \widehat{A}$  which vanishes only for the trivial character  $\mathbf{1}$ , and proves the existence of a meromorphic function  $y_\chi$  and a meromorphic differential  $\psi_\chi$  on  $X$  (denoted by  $h_\chi$  and  $\omega_\chi$  in that reference respectively), on both of which  $A$  operates via  $\chi$ , and such that

$$\operatorname{div}(y_\chi) = \sum_\sigma \sum_{j=1}^{r_\sigma} u_{\chi,\sigma} z^{-1}(\lambda_{\sigma,j}) - t_\chi z^{-1}(\infty)$$

and

$$\operatorname{div}(\psi_\chi) = \sum_\sigma \sum_{j=1}^{r_\sigma} (o(\sigma) - 1 - u_{\overline{\chi},\sigma}) z^{-1}(\lambda_{\sigma,j}) + (t_{\overline{\chi}} - 2) z^{-1}(\infty).$$

One can define  $\psi_\chi$  as  $\frac{dz}{y_{\overline{\chi}}}$  for every  $\chi \in \widehat{A}$ . Note that in this paper we use the additive notation for divisors, rather than not the multiplicative one from [KZ]. From Corollary 5.6 of that reference, combined with the paragraph following it, we deduce the following useful result.

**Proposition 1.2.** *The space of differentials of the first kind on  $X$  decomposes as  $\bigoplus_{\mathbf{1} \neq \chi \in \widehat{A}} \mathcal{P}_{\leq t_{\overline{\chi}}-2}(z) \psi_\chi$ , where  $\mathcal{P}_{\leq d}(z)$  stands for the space of polynomials of degree not exceeding  $d \geq -1$  in  $z$  (with the zero space being  $\mathcal{P}_{\leq -1}(z)$  in this notation).*

We shall also make use of the observations summarized in the following lemma, whose proofs are simple and straightforward.

**Lemma 1.3.** *For any  $\sigma \in A$  and character  $\chi \in \widehat{A}$ , the sum  $u_{\chi,\sigma} + u_{\overline{\chi},\sigma}$  vanishes if  $\chi(\sigma) = 1$  and equals  $o(\sigma)$  otherwise. Therefore the sum  $t_\chi + t_{\overline{\chi}}$  equals  $\sum_{\sigma \notin \ker \chi} r_\sigma$ . Moreover, the divisor of  $y_\chi y_{\overline{\chi}}$  coincides with that of the polynomial  $\prod_{\sigma \notin \ker \chi} \prod_{j=1}^{r_\sigma} (z - \lambda_{\sigma,j})$ .*

The first two assertions of Lemma 1.3 hold equally well for any cover (Abelian or not) of any compact Riemann surface, while the third one also extends to non-Abelian covers of the sphere. For covers of a Riemann surface  $S$  of higher genus, the divisor of  $y_\chi y_{\overline{\chi}}$  only has the property of being a pull-back of a degree 0 divisor on  $S$ .

In the notation of the fibered product, assume that  $\lambda_i$ ,  $1 \leq i \leq s$  contain all the zeros and all the poles of the all the functions  $F_l$ ,  $1 \leq l \leq q$ . We may assume (by replacing  $y_l$  by a scalar multiple) that the  $F_l$ s are quotients of *monic* polynomials, so that each  $F_l(z)$  is of the form  $\prod_{j=1}^s (z - \lambda_j)^{\alpha_{lj}}$ , with  $\alpha_{li} \in \mathbb{Z}$  for every  $1 \leq i \leq s$  and  $m_l \mid \sum_{i=1}^s \alpha_{li}$ . Then Proposition 1.5 of [KZ] shows that if  $\rho_l$  is the generator of  $H_l$  associated with the choice of  $y_l$  and  $\sigma$  is some element

$\prod_{l=1}^q \rho_l^{d_l}$  then the points  $\lambda_{\sigma,j}$  are those points  $\lambda_i$  for which  $\alpha_{li} \in d_l + m_l \mathbb{Z}$  for every  $1 \leq l \leq q$  (and  $r_\sigma$  is the number of these points). For a character  $\chi \in \widehat{A}$  there are numbers  $e_l$ ,  $1 \leq l \leq q$  such that  $\chi(\rho_l) = \mathbf{e}(\frac{e_l}{m_l})$  (and every choice of  $e_l$ s is possible), and we can take

$$y_\chi = \prod_{l=1}^q y_l^{e_l} \Big/ \prod_{i=1}^s (z - \lambda_i)^{\lfloor \sum_{l=1}^q e_l \alpha_{li} / m_l \rfloor},$$

where the symbol  $\lfloor x \rfloor$  stands for the largest integer that does not exceed the real number  $x$ . This expression depends on the image of each  $e_l$  in  $\mathbb{Z}/m_l \mathbb{Z}$  (just like  $\chi$ ), its order at every point in  $f^{-1}(\lambda_i)$  is the same as of  $(z - \lambda_i)^{\lfloor \sum_{l=1}^q e_l \alpha_{li} / m_l \rfloor}$  (using  $\{x\} = x - \lfloor x \rfloor$  for the fractional part of  $x$ ), and  $t_\chi$  is the sum of these fractional parts over  $j$ . With these choices the equality between  $y_\chi y_{\overline{\chi}}$  and the polynomial from Lemma 1.3 holds also as functions (not only on the level of divisors), and actions such as that from Lemma 1.5 below can become, in the appropriate sense described after Lemma 3.3 below, actions on generalized functions (i.e., sections of line bundles) rather than on their divisor.

On the  $Z_{m_l}$ -curve  $Y_l$  associated with  $H_l$  we may normalize the generator  $y_l$  (almost uniquely) such that  $F_l$  becomes a (monic) polynomial containing no  $\lambda_i$  to a power exceeding  $m_l$ . This means that  $y_l$  can be taken as  $y_\chi$  for the character  $\chi$  sending  $\rho_l$  to  $\mathbf{e}(\frac{1}{m_l})$  and the other  $\rho_k$ s (hence the other  $H_k$ s) to 1. If we assume that the branching values of the  $Y_l$ s are mutually disjoint (a condition which is equivalent to  $r_\sigma$  being possibly positive only when  $\sigma$  lies in one of the multipliers  $H_l$ ), an assumption that is made in [Ko3], then we can write  $F_l(z)$  as  $\prod_{i=1}^{s_l} (z - \lambda_{li})^{\alpha_{li}}$  (with  $0 \leq \alpha_{li} < m_l$  for every  $1 \leq i \leq s_l$  such that  $\sum_{i=1}^{s_l} \alpha_{li}$  is divisible by  $m_l$ ), where  $\lambda_{li} \neq \lambda_{kj}$  unless  $l = k$  and  $j = i$ . Then  $\lambda_{li}$  is associated with  $\rho_l^{\alpha_{li}}$ . In this case we define, for each  $0 \leq e < m_l$  (or once again for  $e \in \mathbb{Z}/m_l \mathbb{Z}$ , the function  $y_l^{(e)}$  to be the normalized  $e$ th power of  $y_l$ , where the normalization is  $y_l^e / \prod_{i=1}^{s_l} (z - \lambda_{li})^{e\alpha_{li} - m_l \lfloor e\alpha_{li} / m_l \rfloor}$  as above. Then the basis of differentials from Proposition 1.2 consists of differentials of the form  $z^k dz / \prod_{l=1}^q y_l^{(e_l)}$  for  $q$ -tuples  $\{e_l\}_{l=1}^q$  in which  $0 \leq e_l < m_l$  for every  $l$ , not all the  $e_l$ s vanish, and the bound on  $k$  is  $\sum_{l=1}^q \lfloor \frac{e_l \alpha_{li}}{m_l} \rfloor - 2$ . In the language of Proposition 1.2 the differential with these parameters and with  $k = 0$  is  $\psi_\chi$  where  $\chi$  is the (non-trivial) character in  $\widehat{A}$  that takes  $\rho_l$  to  $\mathbf{e}(-\frac{e_l}{m_l})$  for every  $1 \leq l \leq q$ , since  $\psi_\chi = \frac{dz}{y_{\overline{\chi}}}$  and the disjointness of the branching values implies that  $y_{\overline{\chi}}$  is the product  $\prod_{l=1}^q y_l^{(e_l)}$ .

We shall need  $A$ -invariant divisors  $\Xi$  of degree  $g-1$  on  $X$  that are not linearly equivalent to any integral divisor (adapting the notation from [FK], [FZ], [Z], and [KZ] to our additive notation for divisors, the latter property is the content of the equality  $r(-\Xi) = 0$ ). Proposition 2.4 and Corollary 2.7 of [KZ] allow us to consider, up to multiplication by divisors of rational functions of  $z$ , only divisors of the form

$$\Delta = \sum_{\sigma} \sum_{j=1}^{r_\sigma} \beta_{\sigma,j} f^{-1}(\lambda_{\sigma,j}) - p f^{-1}(\infty), \quad 0 \leq \beta_{\sigma,j} < o(\sigma), \quad p \in \mathbb{Z}. \quad (1)$$

The divisors that we seek, as well as their behavior under the Abel–Jacobi map, are now described in Theorem 4.5 of [KZ], which we now state.

**Theorem 1.4.** *The divisor  $\Delta$  from Equation (1) is of degree  $g - 1$  and satisfies  $r(-\Delta) = 0$  if and only if  $p = 1$  and the equality*

$$\sum_{\sigma} |\{j | \beta_{\sigma,j} \geq o(\sigma) - u_{\chi,\sigma}\}| = t_{\chi}$$

*holds for every  $\chi \in \hat{A}$ .*

To see that the condition in Theorem 1.4 is indeed the one from Theorem 4.5 observe that the set denoted  $B_{\sigma,i}$  in that reference is  $\{j | \beta_{\sigma,j} = o(\sigma) - 1 - i\}$ , and  $0 \leq i \leq u_{\chi,\sigma} - 1$  corresponds precisely to the stated criterion for  $j$ . It will be of value for some calculations to observe that only elements  $\sigma \in A$  for which  $\chi(\sigma) \neq 1$  (i.e.,  $u_{\chi,\sigma} > 0$ ) may effectively contribute to the sum corresponding to  $\chi$  in Theorem 1.4 (in particular the assertion always holds trivially for  $\chi = \mathbf{1}$ ).

Lemma 2.1 of [KZ] incarnates itself in an action of  $\hat{A}$  on the set of divisors from Theorem 1.4. As formulae for the explicit action will later be of use (see Lemma 3.3 below), we present them in the following lemma.

**Lemma 1.5.** *Let  $\Delta$  be a divisor, presented as in Equation (1), such that the conditions of Theorem 1.4 are satisfied, and take an element  $\chi \in \hat{A}$ . If we denote the polynomial  $\prod_{\{j | \beta_{\sigma,j} \geq o(\sigma) - u_{\chi,\sigma}\}} (z - \lambda_{\sigma,j})$  (as a meromorphic function on  $X$ ) by  $p_{\Delta,\chi}$  then the divisor  $\chi\Delta = \Delta + \text{div}(y_{\chi}) - \text{div}(p_{\Delta,\chi})$  is also normalized (i.e., has a presentation as in Equation (1)), and satisfies the conditions of Theorem 1.4 as well. This defines a free action of  $\hat{A}$  on this set of divisors, such that two such divisors are linearly equivalent if and only if they lie in the same  $\hat{A}$ -orbit.*

Lemma 1.5 follows from Lemma 2.1 of [KZ] and the observation that that the parameter associated for  $\sigma$  and  $j$  in  $\chi\Delta$  is  $\beta_{\sigma,j} + u_{\chi,\sigma}$  if  $\beta_{\sigma,j} < o(\sigma) - u_{\chi,\sigma}$  and  $\beta_{\sigma,j} + u_{\chi,\sigma} - o(\sigma)$  otherwise, it lies in the required range. This is the defining reasoning behind the polynomials  $p_{\Delta,\chi}$ . Note that as the degree of  $p_{\Delta,\chi}$  is  $t_{\chi}$  by Theorem 1.4, the order  $-1$  at the points in  $z^{-1}(\infty)$  remains unaffected by the action of  $\chi$ .

## 2 Negating Torsion Theta Characteristics

For any (compact) Riemann surface  $X$ , with canonical homology basis  $a_i$  and  $b_i$ ,  $1 \leq i \leq g$  and resulting symmetric matrix  $\tau$  with positive definite imaginary part, and any vector  $e \in \mathbb{C}^g$ , we denote the associated theta function with characteristics by  $\theta[e]$ . Explicitly,  $e$  has a unique presentation as  $\Pi \frac{\varepsilon}{2} + I \frac{\delta}{2}$  for real vectors  $\varepsilon$  and  $\delta$ , and then

$$\theta[e](z, \tau) = \theta\left[\frac{\varepsilon}{\delta}\right](z, \tau) = \sum_{N \in \mathbb{Z}^g} \mathbf{e}\left[\left(N + \frac{\varepsilon}{2}\right)^t \frac{\tau}{2} \left(N + \frac{\varepsilon}{2}\right) + \left(N + \frac{\varepsilon}{2}\right)^t \left(z + \frac{\delta}{2}\right)\right]$$

with  $\tau$  from above and  $z \in \mathbb{C}^g$ . For the basic properties of theta functions see Chapter 6 of [FK] or Section 1.3 of [FZ].

We consider the Jacobian  $J(X)$  of  $X$  as the quotient of  $\mathbb{C}^g$  modulo the lattice generated by the columns of the identity matrix  $I$  and the columns of  $\tau$ . The Abel–Jacobi map on divisors of degree 0 on  $X$ , which is based on the basis  $v_s$ ,  $1 \leq s \leq g$  for the differentials on the first kind on  $X$  that is dual to the cycles  $a_i$ ,  $1 \leq i \leq g$ , is denoted by  $u$ . We recall that if  $K_R$  is the vector of Riemann constants associated with the base point  $R$  and  $\Delta$  is a divisor of degree  $g - 1$  then the value of  $u_R(\Delta) + K_R = u(\Delta - (g - 1)R) + K_R$  is independent of the choice of  $R$  (see, e.g., Theorem 1.12 of [FZ]). Hence we can write  $u(\Delta) + K$  for such divisors. The Riemann Vanishing Theorem (also mentioned in [FK] and [FZ]) characterizes the zeros of the basic theta function  $\theta(z, \tau) = \theta[0](z, \tau)$  in  $z$  as the pre-images in  $\mathbb{C}^g$  of the points of the form  $u(\Delta) + K$  for integral divisors  $\Delta$  of degree  $g - 1$ . On the other hand, if the function  $\theta[e](u(P - Q))$ , or equivalently  $\theta(u(P - Q) + e)$ , does not vanish identically (as a function of  $Q$ , for fixed  $P$ ) then its divisors  $\Xi_{e,P}$  of zeros is non-special of degree  $g$ , and its image under  $u_P + K_P$  is the image of  $e$  in  $J(X)$  (so that  $e = u(\Xi_{e,P} - P) + K$ , and  $\Xi_{e,P}$  is the unique integral divisor of degree  $g$  satisfying that equality). By the evenness of  $\theta$ , the divisor of zeros of that expression as a function of  $P$  with fixed  $Q$  is  $\Xi_{-e,Q}$ . We shall also be needing the following result, appearing, among others, as Proposition 2.1 of [Z].

**Proposition 2.1.** *If  $\omega$  is any non-zero meromorphic differential on  $X$  and  $R$  is any point on  $X$  then the value  $u_R(\text{div}\omega) = u(\text{div}\omega - (2g - 2)R)$  equals  $-2K_R$ .*

Recall that Theorem 2.3 of [Z] defined, for any point  $Q$  on a compact Riemann surface  $X$  and any non-special integral divisor  $\Xi$  of degree  $g$  not containing  $Q$  in its support, another divisor  $N_Q(\Xi)$  with the same properties such that the elements  $u_Q(\Xi) + K_Q$  and  $u_Q(N_Q(\Xi)) + K_Q$  are inverses. We shall require this negation operation in the language of divisors of degree  $g - 1$ .

**Lemma 2.2.** *If  $\Delta$  is a divisor of degree  $g - 1$  on  $X$  with  $r(-\Delta) = 0$  and  $\omega$  is any non-zero meromorphic differential on  $X$  then  $\Gamma = \text{div}\omega - \Delta$  also has these properties, and the values  $u(\Delta) + K$  and  $u(\Gamma) + K$  are additive inverses. If  $Q$  is any point on  $X$  then there is a unique integral divisor  $\Xi_{\Delta,Q}$  of degree  $g$ , depending only on the linear equivalence class of  $\Delta$ , that is linearly equivalent to  $\Delta + Q$ , and this divisor does not contain  $Q$  in its support. Moreover,  $\Xi_{\Gamma,Q}$  is defined similarly, it is independent of the choice of  $\omega$ , and it equals  $N_Q(\Xi_{\Delta,Q})$ .*

*Proof.* We have  $\deg \Gamma = g - 1$  since canonical divisors have degree  $2g - 2$ . Moreover, all the divisors that are linearly equivalent to  $\Gamma$  are obtained from different choices of  $\omega$ . As none of these divisors can be integral (since otherwise we get a contradiction to the condition  $r(-\Delta) = 0$ ), the condition  $r(-\Delta) = 0$  follows as well. As the sum of  $u_P(\Delta) + K_P$  and  $u_P(\Gamma) + K_P$  (where  $P$  is any base point) is  $u_P(\text{div}\omega) + 2K_P$  and hence vanishes by Proposition 2.1, the first assertion is established. Next, if  $r(-\Delta) = 0$  and  $\deg \Delta = g - 1$  then the



Riemann–Roch Theorem implies that there is no meromorphic differential on  $X$  such that subtracting  $\Delta$  from its differential yields an integral divisor (this is written as  $i(\Delta) = 0$  in the notation of [FK] and others). Hence  $i(\Delta + Q) = 0$  as well, and another application of Riemann–Roch yields  $r(-\Delta - Q) = 1$ . This implies the existence and uniqueness of the integral divisor  $\Xi_{\Delta, Q}$  (whose degree is clearly  $g$ ), and it cannot contain  $Q$  in its support since otherwise the same meromorphic function with divisor  $\Xi_{\Delta, Q} - \Delta - Q$  would contradict the fact that  $r(-\Delta) = 0$ . The independence of this construction under replacing  $\Delta$  by a linearly equivalent divisor is obvious, and we also deduce that  $N_Q(\Xi_{\Delta, Q})$  is defined. As replacing  $\omega$  by another meromorphic differential takes  $\Gamma$  to a linearly equivalent divisor, the independence of  $\Xi_{\Gamma, Q}$  also follows. Finally, since adding  $u_Q(\Xi_{\Delta, Q}) + K_Q$  and  $u_Q(\Xi_{\Gamma, Q}) + K_Q$  yields (by linear equivalence) the same value as the sum of  $u_Q(\Delta) + K_Q$  and  $u_Q(\Gamma) + K_Q$ , which is 0, the last assertion follows from the uniqueness assertion in Theorem 2.3 of [Z]. This proves the lemma.  $\square$

The divisor  $\Xi_{e, P}$  from above is precisely  $\Xi_{\Delta, P}$  from Lemma 2.2, when  $e$  and  $\Delta$  are related via the equality  $e = u(\Delta) + K$ . Its  $N_P$ -image is therefore  $\Xi_{-e, P}$ .

We are interested in the negation operator from Lemma 2.2 for our case, where there is a map  $z : X \rightarrow \mathbb{CP}^1$  making  $X$  an Abelian cover of the sphere with Galois group  $A$ . Moreover, we will be interested in  $A$ -invariant divisors, for which we can reduce attention (by Proposition 2.4 and Corollary 2.7 of [KZ]) only to those divisors presented in Equation (1). The result is just a corollary of the proof of Lemma 2.2.

**Corollary 2.3.** *If  $\Delta$  is a divisor written as in Equation (1) and satisfying the conditions of Theorem 1.4 then an  $A$ -invariant representative for the class complementary class from Lemma 2.2 can be taken as  $\text{div}(dz) - \Delta$ .*

Indeed, the proof of Lemma 2.2 allows us to take a complement of  $\Delta$  to a canonical divisor of our choice. But with the choice from Corollary 2.3, which we shall henceforth denote by  $N\Delta$ , we obtain a divisor that also takes the form from Equation (1) (the parameter for  $\sigma$  and  $j$  is  $o(\sigma) - 1 - \beta_{\sigma, j}$ ), and the conditions from Theorem 1.4 are also satisfied. This either follows from that theorem, or can be seen directly: For the orders at the poles of  $z$  it is clear (since  $dz$  has order  $-2$  at every such point), and one can easily verify that the polynomial  $p_{N\Delta, \bar{\chi}}$  is the monic one whose zeros are precisely those  $\lambda_{\sigma, j}$  with  $\sigma \notin \ker \chi$  and such that  $p_{\Delta, \chi}$  does not vanish on pre-images of  $\lambda_{\sigma, j}$  (recall that  $p_{\Delta, \chi}$  is considered as a function on  $X$ , not of  $z \in \mathbb{CP}^1$ ). This means that  $p_{\Delta, \chi} p_{N\Delta, \bar{\chi}}$  equals the polynomial whose divisor coincides with that of  $y_{\chi} y_{\bar{\chi}}$  by Lemma 1.3. We remark that Corollary 2.3 holds for any normalized divisor  $\Delta$  of degree  $g - 1$  with  $r(-\Delta) = 0$  on any Galois cover of  $\mathbb{CP}^1$  (not necessarily Abelian) by the same argument (yielding a normalized  $N\Delta$ ), but there is no canonical choice for covers (Abelian or not) of higher genus curves because of the divisors denoted  $\Upsilon$  in [KZ] (for a good example of this, consider  $X$  as a trivial cover of itself).

The relation between the actions of  $\hat{A}$  and  $N$  is as follows.

**Corollary 2.4.** *The actions of  $N$  and  $\hat{A}$  on the set of divisors  $\Delta$  from Theorem 1.4, given in Corollary 2.3 and Lemma 1.5 respectively, satisfy the equality  $N\chi\Delta = \bar{\chi}N\Delta$  for every such  $\chi$  and  $\Delta$ . Therefore  $N$  and  $\hat{A}$  generate a generalized dihedral group of operators acting on this set of divisors.*

This follows directly from the formulae for the actions given in Corollary 2.3 and Lemma 1.5 and the coincidence of the divisors of  $y_\chi y_{\bar{\chi}}$  and  $p_{\Delta,\chi} p_{N\Delta,\bar{\chi}}$ . Corollary 2.4 generalizes Lemma 5.2 of [Z] from the cyclic case (including our notation  $N$ ). On characteristics (i.e., images in  $J(X)$  under  $u + K$ ) this action reduces to an action of the 2-cyclic quotient generated by  $N$  operating just as  $\{\pm 1\}$ . Note that in the language of integral divisors of degree  $g$ , the fact that at the end of Section 3 of [Z] the notation for the negation operator was dependent on the index  $\beta$  is a reminiscent of the fact that the divisors come in classes that are  $\hat{A}$ -orbits.

We remark that while we were successful in generalizing the operator  $N$  (or  $N_Q$ ) from [Z] to this much more general setting, for the other types of operators defined in the reference, those denoted by  $T_{Q,R}$  (or its simpler versions  $\hat{T}_{Q,R}$ ) a generalization do not extend so directly. This is because the definition of these operators used the existence of a single,  $A$ -invariant point on  $X$  lying over any branching value in  $\mathbb{CP}^1$  in the fully ramified cyclic case, a situation the occurs under no other assumptions. This is also the reason why the elementary methods of [FZ] and [Z] (and some references cited there) for proving Thomae type formulae probably require a more delicate argument in more general cases.

Recall that by Lemma 2.2 of [Z], the characteristic  $u(\Delta) + K$  arising from any  $A$ -invariant divisor  $\Delta$  of degree  $g - 1$  (regardless of the value of  $r(-\Delta)$ ) is, in the fully ramified cyclic case, torsion in  $J(X)$ , of order dividing  $2n$ . While we shall not use this result in our arguments, it is good to know that this property holds in our more general situation as well. We denote by  $m$  the exponent of  $A$ , which equals, given a factorization of  $A$  as the direct product of the cyclic subgroups  $H_l$ ,  $1 \leq l \leq q$  with  $|H_l| = m_l$ , their least common multiple  $\text{lcm}\{m_l\}_{l=1}^q$  (this is just  $m_1$  in case the  $H_l$ s are normalized such that  $m_l | m_{l-1}$  for any  $l > 1$ ).

**Proposition 2.5.** *If  $\Delta$  is an  $A$ -invariant divisor of degree  $g - 1$  then the characteristic  $u(\Delta) + K$  is torsion of order dividing  $2\text{lcm}\{m, \frac{n}{m}\}$ .*

*Proof.* Take sum non-trivial element  $\rho \in A$  and some  $1 \leq i \leq r_\rho$ , and then write  $\frac{n}{o(\rho)}(u(\Delta) + K)$  as  $\sum_{Q \in f^{-1}(\lambda_{\rho,i})} (u_Q(\Delta) + K_Q)$  (denoting these points  $Q$  as in [KZ], this sum becomes  $\sum_{v=1}^{n/o(\rho)} (u_{P_{\rho,i,v}}(\Delta) + K_{P_{\rho,i,v}})$ ). We show that both the terms with  $u_Q(\Delta)$  and the constants  $K_Q$  sum to torsion points, beginning with the former. As  $\Delta$  is the sum of expressions of the forms  $\pm z^{-1}(\lambda)$  with  $\lambda \in \mathbb{CP}^1$ , it suffices to prove this claim for one such expression, with  $\pm = +$ . Such an expression is of degree  $\frac{n}{o(\sigma)}$  in case  $\lambda = \lambda_{\sigma,j}$  for some  $j$ , a claim which extends to  $\lambda$ s which are not branch values since such elements of  $\mathbb{CP}^1$  are mapped to the identity element of  $G$  by the map from Proposition 1.1 of [KZ]. Thus  $u_Q(z^{-1}(\lambda))$  is  $u(z^{-1}(\lambda) - \frac{n}{o(\rho)}Q)$ , and the sum over  $Q$  equals  $u(\frac{n}{o(\rho)}z^{-1}(\lambda) - \frac{n}{o(\sigma)}z^{-1}(\lambda_{\rho,i}))$ . We now observe that the function  $\frac{z-\lambda}{z-\lambda_{\rho,i}}$  has divisor  $o(\sigma)z^{-1}(\lambda) - o(\rho)z^{-1}(\lambda_{\rho,i})$ ,

and the argument of  $u$  is the rational multiple of this divisor, the multiplier being  $\frac{n}{o(\sigma)o(\rho)}$ . Multiplying that number by  $\frac{o(\sigma)o(\rho)}{\gcd\{n, o(\sigma)o(\rho)\}}$  yields an integer, so that multiplying the latter  $u$ -value by that number, which we can write as  $o(\rho)/\gcd\{\frac{n}{o(\sigma)}, o(\rho)\}$ , yields 0. As the latter expression increases with  $o(\sigma)$ , replacing  $o(\sigma)$  by its maximal possible value  $m$  works for every choice of  $\lambda$  hence for every  $A$ -invariant divisor  $\Delta$  as well.

We therefore proved that  $\sum_{Q \in f^{-1}(\lambda_{\rho,i})} u_Q(\Delta)$  is a point of order dividing  $o(\rho)/\gcd\{\frac{n}{m}, o(\rho)\}$  wherever  $\Delta$  is  $A$ -invariant. But  $-2K_Q$  is also the  $u_Q$ -image of an  $A$ -invariant divisor: Indeed, every canonical  $A$ -invariant divisor, such as  $\text{div}(dz)$ , maps to that value. Therefore  $\sum_{Q \in f^{-1}(\lambda_{\rho,i})} K_Q$  is torsion of order dividing  $2o(\rho)/\gcd\{\frac{n}{m}, o(\rho)\}$ . Recalling that the sum of these two expressions is  $\frac{n}{o(\rho)}(u(\Delta) + K)$ , we deduce that our characteristic of interest has order dividing  $2n/\gcd\{\frac{n}{m}, o(\rho)\}$ . We now observe that this order can be written as  $\frac{2nm}{o(\rho)}/\gcd\{m, \frac{n}{o(\rho)}\}$ , which by the relation between the gcd and lcm of two numbers equals  $2\text{lcm}\{m, \frac{n}{o(\rho)}\}$ . But the order of  $u(\Delta) + K$  cannot depend on  $\rho$ , and latter order decreases with  $o(\rho)$ . Therefore we get the strongest assertion when  $o(\rho)$  also attains the maximal value  $m$ , which completes the proof of the proposition.  $\square$

Note that Proposition 2.5 combines with Lemma 1.5 to show that the set of divisors satisfying the conditions of Theorem 1.4 is finite, in correspondence with Corollary 3.3 of [KZ] (this is also clear from the form of Equation (1) and the condition  $p = 1$  in the latter theorem). In fact, Proposition 3.2 below implies that the order of these characteristics must be a divisor of the smaller number  $2m$ . In some cases, like the case of non-singular  $Z_n$  curves with even  $n$  (see Chapter 4 of [FZ]) or the case with  $m = 2$  considered in [Ko3], one can even reduce the bound order to  $m$ . For the context of [KZ] we remark that the proof of Proposition 2.5 does not depend on  $A$  being Abelian, hence holds for every cover of  $\mathbb{CP}^1$ . For a more general Galois cover  $f : X \rightarrow S$  (Abelian or not), pre-images of divisors on  $S$  enter the calculations (both in the formula for  $\Delta$  and in the fact that the equivalent of the divisor  $\lambda - \lambda_{\rho,i}$  of the meromorphic function  $\frac{z-\lambda}{z-\lambda_{\rho,i}}$  becomes a divisor of degree 0 on  $S$  which is not principal in general). In this case Proposition 2.5 shows that the image of  $u(\Delta) + K$  in the Prym variety  $P(X/S)$  complementing the image of  $f^*J(S)$  inside  $J(X)$  is torsion of the asserted order.

### 3 An Expression for The Szegő Kernel

Given a compact Riemann surface  $X$  (with  $\tau$  and  $u$  as above) and a point  $e \in \mathbb{C}^g$  with  $\theta[e](0, \tau) \neq 0$ , the *Szegő kernel associated with  $e$*  is defined by the equation

$$S[e](P, Q) = \frac{\theta[e](u(P - Q), \tau)}{\theta[e](0, \tau)E(P, Q)}, \quad \text{with } P \text{ and } Q \text{ in } X.$$

Here  $E(P, Q)$  is the *prime form*, an anti-symmetric holomorphic  $(-\frac{1}{2}, -\frac{1}{2})$ -form that vanishes only on the diagonal  $P = Q$ , such that its expansion in  $z(P)$  around  $z(Q)$  in a coordinate chart  $z$  is  $\frac{z(P)-z(Q)}{\sqrt{dz(P)}\sqrt{dz(Q)}} \left[ 1 + O\left((z(P)-z(Q))^2\right) \right]$ . Some useful properties of  $S[e]$  are given in the following proposition, a proof of which one can find in [Fa] (see pages 19 and 123 there), [EG1] (see Sections 3 and 4, in particular the proof of Theorem 4.7), or [Na], but it also follows from direct calculations and the considerations from the previous paragraph.

**Proposition 3.1.**  *$S[e](P, Q)$  is a  $(\frac{1}{2}, \frac{1}{2})$ -form which depends only on the image of  $e$  in  $J(X)$ . Its divisor in  $Q$  for fixed  $P$  is  $\Xi_{e,P} - P$ , while for fixed  $Q$  its divisor in  $P$  is  $\Xi_{-e,Q} - Q$ . In both variables it is the section of the line bundle of degree  $g-1$  corresponding to  $e$  that transforms according to unique associated unitary character of the homology of  $X$ : Adding a cycle  $a_i$  or  $b_i$  to  $P$  (resp.  $Q$ ) multiplies the value of  $S[e](P, Q)$  by  $\mathbf{e}(\frac{\varepsilon_i}{2})$  or  $\mathbf{e}(-\frac{\delta_i}{2})$  (resp. their inverses) when  $e = \Pi_{\frac{\varepsilon}{2}} + I_{\frac{\delta}{2}}$ . In particular, it is holomorphic on  $X \times X$  except for a simple pole along the diagonal, the expansion around which in a coordinate  $z$  as above is of the form*

$$\frac{\sqrt{dz(P)}\sqrt{dz(Q)}}{z(P)-z(Q)} \left[ 1 + \sum_{s=1}^g \frac{\partial \ln \theta[e]}{\partial z_s} \Big|_{z=0} \frac{v_s(Q)}{dz(Q)} (z(P)-z(Q)) + O\left((z(P)-z(Q))^2\right) \right].$$

Finally, it is unique  $(\frac{1}{2}, \frac{1}{2})$ -form that satisfies these properties.

The only part of Proposition 3.1 that does not follow directly from basic properties of the theta functions or of the prime form is the last assertion. But it is proved in [Na], [Ko2], [Ko3], and others by observing the fact that the line bundle on  $X \times X$  of which  $S[e]$  is a section is the tensor product of pull-backs of line bundles on the copies of  $X$  (this is because the divisors  $\Xi_{e,P} - P$  for the different points  $P$  are linearly equivalent, having the same degree  $g-1$  and the same image  $e$  under  $u + K$ , and the same for  $\Xi_{-e,Q} - Q$ ). The difference of two such forms is thus a holomorphic section of that bundle, and space of holomorphic sections there is the external tensor product of the spaces of holomorphic sections on the two line bundles on  $X$ . But those line bundles have no holomorphic global sections (since the fact that  $\theta[e](0, \tau) \neq 0$  implies that no integral divisor is linearly equivalent to any  $\Xi_{e,P} - P$  or to any  $\Xi_{-e,Q} - Q$ ), so that the difference in question is 0. This is just the argument completing the proofs of Theorem 1.1 of [Na], Theorem 7.2 of [Ko2], and Theorem 6.2 of [Ko3], but it holds equally well for the Szegő kernel on any Riemann surface (regardless of a Galois cover structure). We also remark that the expansion of both  $E(P, Q)$  and  $S_e(P, Q)$  around the diagonal transforms well under coordinate changes, and therefore holds equally well for every coordinate chart.

Our goal is to generalize the approach of [Na] and construct  $S[e](P, Q)$ , where  $e$  is  $u(\Delta) + K$  for  $\Delta$  one of the divisors satisfying the conditions of Theorem 1.4, algebraically for our type of general Abelian covers. Let now  $\Delta$  be one of the divisors satisfying the conditions of Theorem 1.4, write it as in Equation (1),

and consider the expression

$$f_\Delta(P) = \prod_{\sigma} \prod_{j=1}^{r_\sigma} (z(P) - \lambda_{\sigma,j})^{\frac{\beta_{\sigma,j}}{o(\sigma)} - \frac{o(\sigma)-1}{2o(\sigma)}} \cdot \sqrt{dz(P)}, \quad P \in X. \quad (2)$$

We now prove a generalization of Proposition 4 of [Na], which is also implicitly used in [Ko2] and [Ko3].

**Proposition 3.2.** *The expression  $f_\Delta$  from Equation (2) is a well-defined meromorphic section, with divisor  $\Delta$ , of the line bundle of degree  $g-1$  on  $X$  that is associated with  $u(\Delta) + K$  and whose transformation rule under the homology is according to the unique associated unitary character.*

*Proof.* We begin by observing the none of the multipliers is well-defined on  $X$ , all of them do make sense on an appropriate finite cover  $\tilde{X}$  of  $X$ . Recalling that  $z - \lambda_{\sigma,j}$  has order  $o(\sigma)$  at any pre-image of  $\lambda_{\sigma,j}$  while  $dz$  has the order  $o(\sigma) - 1$  there, we find that if  $f_\Delta$  is to be well-defined locally at that pre-image, its order there would be  $\beta_{\sigma,j}$  (just like its multiplicity in  $\Delta$ ). Consider now a small neighborhood  $V$  of that pre-image in  $X$ , and take a branch of  $f_\Delta$  that is defined on a pre-image  $\tilde{V}$  of  $V$  in  $\tilde{X}$ . The fact that the aforementioned order is integral implies that if we take a path in  $\tilde{V}$  that maps to a closed path in  $V$  then  $f_\Delta$  attains the same value at the two end points. Therefore  $f_\Delta$  can be considered as well-defined on  $V$  (equivalently, analytically continuing a branch of  $f_\Delta$  at a small open set in  $V$  not mapping onto  $\lambda_{\sigma,j}$  to all of  $V$  is possible, since the values of the analytic continuation do not depend on the path one takes inside  $V$ ). For open sets in  $X$  not containing any  $\lambda_{\sigma,j}$  neither the poles of  $z$  the well-definedness is immediate from the definition, as well as the fact that these sections have neither zeros nor poles in such sets. At the poles of  $z$  the expression  $\sqrt{dz}$  has a simple pole (since  $dz$  itself has a double pole), and when we add the contributions from the other we get another pole of order

$$\sum_{\sigma} \sum_{j=1}^{r_\sigma} \left( \frac{\beta_{\sigma,j}}{o(\sigma)} - \frac{o(\sigma)-1}{2o(\sigma)} \right) = \frac{\deg \Delta + n}{n} - \frac{g + n - 1}{n}$$

by the expression for  $\Delta$  in Equation (1) and the formula for  $g$  appearing in Proposition 1.1. But as Theorem 1.4 implies that the degree of  $\Delta$  is  $g-1$ , the latter difference vanishes, and we get simple poles for  $f_\Delta$  at the poles of  $z$  (again, just like in  $\Delta$ ). The same argument from above shows that  $f_\Delta$  is well-defined locally also around these points.

Since  $f_\Delta$  is well-defined locally at every point of  $X$ , connecting the local definitions makes  $f_\Delta$  a meromorphic section (recall the poles at  $z^{-1}(\infty)$ ) of a holomorphic line bundle on  $X$ . Knowing the divisor  $\Delta$  of  $f_\Delta$  determines the isomorphism class of this line bundle, but we are interested in its explicit presentation. However, since  $f_\Delta$  is defined using fractional powers of polynomials in  $z$  and  $dz$ , with the denominators dividing  $2m$ , the expression  $f_\Delta^{2m}$  is well-defined on  $X$ . Therefore the ambiguity in  $f_\Delta$  is only via multiplication by roots of unity

of order  $2m$ , so that in particular going over a homology cycle can only multiply  $f_\Delta$  by a complex number of absolute value 1. Since this presentation of any line bundle is unique, this proves the proposition.  $\square$

In fact, the proof of Proposition 3.2 also implies the result that the characteristic  $u(\Delta) + K$  is torsion of order  $2m$ . Indeed, it proves that  $f_\Delta$  is a section of the line bundle associated with  $\Delta$  (of degree  $g - 1$ ), and the line bundle of which  $dz$  is a section is associated with  $-2K$  (of degree  $2g - 2$ ) by Proposition 2.1. Therefore for a section of the degree 0 line bundle associated with  $2m(u(\Delta) + K)$  we can take the quotient  $\frac{f_\Delta^{2m}}{(dz)^m}$ . But this is just a rational function on  $z$ , which is meromorphic on  $X$  hence the line bundle of which it is a section is a trivial one.

**Lemma 3.3.** *Given  $f_\Delta$  and  $\chi \in \widehat{A}$ , the product  $\frac{y_\chi}{p_{\Delta, \chi}} f_\Delta$ , which is a meromorphic section of the same line bundle as  $f_\Delta$ , is a scalar multiple of  $f_{\chi\Delta}$  for every  $\chi \in \widehat{A}$ . Moreover,  $\frac{dz}{f_\Delta}$  is a scalar multiple of  $f_{N\Delta}$ .*

*Proof.* The fact that  $f_{\chi\Delta}$  and  $f_\Delta$  are meromorphic sections of the same line bundle follows from Proposition 3.2 and Lemma 1.5, and that lemma also shows that the product in question has the required divisor  $\chi\Delta$ . This proves the first assertion. As for the second one, the fact that  $N$  sends every  $\beta_{\sigma, j}$  in Equation (1) to  $o(\sigma) - 1 - \beta_{\sigma, j}$  implies that it inverts the powers appearing in Equation (2). Therefore (up to the root of unity ambiguity) all the terms with  $z - \lambda_{\sigma, j}$  cancel in the product  $f_\Delta f_{N\Delta}$ , and only  $dz$  remains (this also follows, though slightly less directly, from Propositions 2.2 and 3.2 and Corollary 2.3). This proves the lemma.  $\square$

The generalization of the algebraic construction of the Szegő kernel appearing in [Na], [EG2], [Ko2], and [Ko3], in which  $e$  is the image of one of the divisors from Theorem 1.4 under  $u + K$ , is dividing the sum  $\sum_{\chi \in \widehat{A}} f_{\chi\Delta}(P) f_{N\chi\Delta}(Q)$  by  $z(P) - z(Q)$  (multiplied by  $n$ ). Indeed, this sum is over all the divisors mapping to  $e$ , combined with the pre-image of  $-e$  in a canonical way, so that it is independent of the choice of the pre-image  $\Delta$  of  $e$  under  $u + K$ . However, an issue that has to be clarified, and is slightly overlooked in these references, is that the expressions  $f_{\chi\Delta}$  etc. can be replaced by their scalar multiples, and this does affect the properties of the resulting expressions. Indeed, by letting every  $f_{\chi\Delta}(P)$  and every  $f_{N\chi\Delta}(Q)$  be multiplied by a different scalar, the properties of the aforementioned sum will generally change (we shall see in the proof of Theorem 3.4 below precisely where this point is important). Using the functions  $y_\chi$  appearing in Lemma 3.3 does not solve this problem, since these are also well-defined only up to scalar multiples. One way to do this is to use the choices of the  $y_\chi$  using the fibered product structure on  $X$ , thus producing a well-defined action of  $\widehat{A}$  on the functions of the form  $f_\Delta$  themselves (lying over the action from Lemma 1.5 on their divisors), and by letting  $N$  act as in Lemma 3.3 as well we get an action of the generalized dihedral group from Corollary 2.4 on

these functions as well. However, a weaker normalization suffices for obtaining the required result.

**Theorem 3.4.** *Assume that  $e = u(\Delta) + K$  for some of the divisors  $\Delta$  from Theorem 1.4, and define  $f_\Delta$  as in Equation (2). Given explicit choices of the functions  $y_\chi$  with  $\chi \in \widehat{A}$ , set  $f_{\chi\Delta}^+$  to be the product  $\frac{y_\chi}{p_{\Delta,\chi}} f_\Delta$  from Lemma 3.3, and define  $f_{N\chi\Delta}^+$  as  $\frac{dz}{f_{\chi\Delta}^+}$ . Then the expression*

$$F_e(P, Q) = \frac{1}{n} \frac{\sum_{\chi \in \widehat{A}} f_{\chi\Delta}^+(P) f_{N\chi\Delta}^+(Q)}{z(P) - z(Q)}, \quad \text{for } P \text{ and } Q \text{ in } X,$$

*is invariant under replacing  $\Delta$  by another divisor mapping to  $e$  and under scalar multiplications of the  $y_\chi$ s (as long as the product  $f_{\chi\Delta}^+ f_{N\chi\Delta}^+$  remain  $dz$  with no additional scalar), and it coincides with the value of the Szegő kernel  $S[e](P, Q)$ .*

Note that  $f_\Delta^+$  itself may not equal  $\Delta$ , since  $y_1$  can be an arbitrary non-zero scalar. Moreover, in general there is no ambiguity in the definition of the expression  $F_e(P, Q)$  here, since we start with  $\Delta$  and define all the  $f_{\chi\Delta}^+$ s and later all the  $f_{N\chi\Delta}^+$ s uniquely. The only restriction here is where  $e$  has order 2 (or 1) in  $J(X)$ , where the action of the dihedral group from Corollary 2.4 is not free. But this case can still give well-defined expressions using the more delicate normalization appearing before Theorem 3.4.

*Proof.* Replacing  $y_\chi$  by  $cy_\chi$  for  $c \in \widehat{C}^\times$  multiplies  $f_{\chi\Delta}^+$  by  $c$ , but also divides  $f_{N\chi\Delta}^+$  by  $c$  since their product remains  $\sqrt{dz}$  by our assumption. Therefore each of the summands is independent of scalar multiplications (as long as the product remains  $dz$ ), and therefore the initial choice of the pre-image  $\Delta$  of  $e$  also has no effect on the value of  $F_e(P, Q)$ . The fact that  $F_e$  is a section of the same line bundle on  $X \times X$  as  $S[e]$  follows immediately from Proposition 3.2 and the uniqueness in Proposition 3.1. Therefore it suffices to show that  $F_e$  is holomorphic outside the diagonal in  $X \times X$  and has the required expansion.

Now, the only poles of the sum over  $\chi$  is at the poles of  $z$ . Therefore wherever  $P$  and  $Q$  are points in  $X$  such that  $z(P)$  and  $z(Q)$  are distinct finite numbers,  $F_e$  is holomorphic at  $(P, Q)$ . When one of these values is infinite (but not the other), the cancelation of the simple poles of the expressions  $f_{\chi\Delta}^+$  or  $f_{N\chi\Delta}^+$  with that of  $z(P) - z(Q)$  also proves the required holomorphicity. For investigating the behavior of  $F_e(P, Q)$  where  $P$  is in the neighborhood of  $\sigma Q$  for some  $\sigma \in A$ , let us fix  $Q$  with finite  $z$ -value that does not equal  $\lambda_{\rho,i}$  for any  $\rho \in A$  and  $1 \leq i \leq r_\rho$  (so that  $z$  is a good coordinate around  $Q$ ), and recall that by our definition the ratio  $\frac{f_{\chi\Delta}^+}{f_\Delta}$  is  $\frac{y_\chi}{p_{\Delta,\chi}}$ . Setting  $f_{N\Delta} = \frac{dz}{f_\Delta}$  as well, the fact that the products  $f_\Delta f_{N\Delta}$  and  $f_{\chi\Delta}^+ f_{N\chi\Delta}^+$  coincide implies that  $\frac{f_{N\chi\Delta}^+}{f_{N\Delta}}$  is the inverse quotient  $\frac{p_{\Delta,\chi}}{y_\chi}$  (this does not necessarily equal  $\frac{y_\chi}{p_{N\Delta,\chi}}$ , as it may be a non-trivial scalar multiple of it). We

can therefore write

$$\sum_{\chi \in \hat{A}} f_{\chi\Delta}^+(P) f_{N\chi\Delta}^+(Q) = \left[ \sum_{\chi \in \hat{A}} \frac{y_\chi(P)}{y_\chi(Q)} \frac{p_{\Delta,\chi}(Q)}{p_{\Delta,\chi}(P)} \right] f_\Delta(P) f_{N\Delta}(Q),$$

and we expand it around  $P = \sigma Q$  using  $z$  as the local coordinate, recalling that  $z(\sigma Q) = z(Q)$ . This means that in the expression corresponding to  $\chi$  inside the brackets we can write  $y_\chi(P) = y_\chi(\sigma Q) + O(z(P) - z(Q))$ , and since  $p_{\Delta,\chi}$  is finite and non-zero at  $\sigma Q$  (recall that it is a polynomial in  $z$  having only branching values as roots), we can write  $\frac{1}{p_{\Delta,\chi}(P)}$  as  $\frac{1}{p_{\Delta,\chi}(\sigma Q)} + O(z(P) - z(Q))$  as well. But as  $p_{\Delta,\chi}$  is a function of  $z$ , it attains the same value on  $Q$  and on  $\sigma Q$ , and since  $A$  operates on  $y_\chi$  via  $\chi$  (recall that  $y_\chi$  is an element of the space denoted  $\mathbb{C}(X)_\chi$  in [KZ]), the ratio  $\frac{y_\chi(\sigma Q)}{y_\chi(Q)}$  is just  $\chi(\sigma)$ . Therefore the sum in the brackets reduces to  $\sum_{\chi \in \hat{A}} \chi(\sigma) + O(z(P) - z(Q))$ , and the latter part is known to equal  $n$  if  $\sigma$  is trivial and 0 otherwise (this is the point where the normalization is used—without it the ratio between  $f_{\chi\Delta}^+(P) f_{N\chi\Delta}^+(Q)$  and  $f_\Delta(P) f_{N\Delta}(Q)$  will be some constant  $c_\chi$  times the aforementioned expression, which yields around  $P = \sigma Q$  the sum  $\sum_{\chi \in \hat{A}} c_\chi \chi(\sigma)$ , and the only situation where this sum vanishes for non-trivial  $\sigma$  and yields  $n$  for the trivial one is when all the  $c_\chi$ s equal 1). Dividing by  $n$  and by  $z(P) - z(Q)$ , we get a holomorphic expression if  $\sigma$  is non-trivial (i.e., when  $P \neq Q$ ), and an expansion of the sort  $\frac{f_\Delta(P) f_{N\Delta}(Q)}{z(P) - z(Q)} [1 + O(z(P) - z(Q))]$  when  $P$  is in the neighborhood of  $Q$ . Since  $\frac{f_{N\Delta}}{\sqrt{dz}}$  was seen to be the inverse of  $\frac{f_\Delta}{\sqrt{dz}}$ , and  $\frac{f_\Delta(P)}{\sqrt{dz(P)}}$  can be written as  $\frac{f_\Delta(Q)}{\sqrt{dz(Q)}} + O(z(P) - z(Q))$ , an additional cancelation produces the desired expansion  $\frac{\sqrt{dz(P)} \sqrt{dz(Q)}}{z(P) - z(Q)} [1 + O(z(P) - z(Q))]$ . The holomorphicity at the points  $(\sigma Q, Q)$  where  $\sigma$  is non-trivial and  $z(Q)$  is  $\infty$  or some point  $\lambda_{\sigma,j}$  now follows from the fact that they are isolated points around which  $F_e$  is holomorphic, and the pole at the points  $(Q, Q)$  for such  $Q$  with the required expansion (modified to use a local coordinate) follows by continuity and the invariance of this expansion under changing the coordinate (though these cases can also be verified directly using a similar calculation as well). This completes the proof of the theorem.  $\square$

In the non-singular  $Z_n$  curve case [Na] uses the next step of the expansion around  $P = Q$  in order to prove Theorem 3.4 for his case, and [Ko2] and [Ko3] also use the following argument. This allows them to deduce the vanishing of all the theta derivatives of the first order at any of the points  $e$  considered in their versions of Theorem 3.4. We now prove that this is the case in general.

**Proposition 3.5.** *For any characteristic  $e$  appearing in Theorem 3.4, all the derivatives  $\frac{\partial \theta[e]}{\partial z_s} \Big|_{z=0}$  with  $1 \leq s \leq g$  vanish.*

*Proof.* Recalling that these derivative appear in the expansion of  $S[e](P, Q)$  around  $P = Q$  in general (see Proposition 3.1), Theorem 3.4 allows us to consider the expansion of  $F_e(P, Q)$  around  $P = Q$  instead in order to obtain information



about them. Once again we shall fix  $Q$  that is neither in  $z^{-1}(\infty)$  nor in any  $z^{-1}(\lambda_{\sigma,j})$ , so that all the terms  $f_{\chi\Delta}^+$  are holomorphic and non-vanishing around  $Q$  and we can use  $z$  as a local coordinate there. For simplicity let us consider the expansion of the part

$$\sum_{\chi \in \hat{A}} f_{\chi\Delta}^+(P) f_{N\chi\Delta}^+(Q) = \left[ \sum_{\chi \in \hat{A}} \frac{f_{\chi\Delta}^+(P)}{\sqrt{dz(P)}} \cdot \frac{f_{N\chi\Delta}^+(Q)}{\sqrt{dz(Q)}} \right] \sqrt{dz(P)} \sqrt{dz(Q)},$$

and the summand in the brackets can be written, for every  $\chi$ , as  $\frac{f_{\chi\Delta}^+(P)}{\sqrt{dz(P)}} / \frac{f_{\chi\Delta}^+(Q)}{\sqrt{dz(Q)}}$  because of our assumption on the normalization between  $f_{\chi\Delta}^+$  and  $f_{N\chi\Delta}^+$ . Expanding this expression around  $P = Q$  using the coordinate  $z$ , the constant term (depending only on  $Q$ ) is 1, and the element of the first order is

$$\left\{ \left[ \frac{d}{dP} \frac{f_{\chi\Delta}^+(P)}{\sqrt{dz(P)}} \right]_{P=Q} \right\} / \frac{f_{\chi\Delta}^+(Q)}{\sqrt{dz(Q)}} \times (z(P) - z(Q)).$$

But the coefficient of  $z(P) - z(Q)$  is thus the derivative of  $\ln \frac{f_{\chi\Delta}^+}{\sqrt{dz}}$  evaluated at  $Q$  (which is well-defined by our assumption on  $z(Q)$ ), an expression which is invariant under scalar multiplication of  $f_{\chi\Delta}^+$  and of the choice of branch for the logarithm. Therefore we can replace  $f_{\chi\Delta}^+$  by the expression for  $f_{\chi\Delta}$  given in Equation (2), using the parameters for the formula for  $\chi\Delta$  in Lemma 1.5. For  $\chi = \mathbf{1}$  the resulting formula is  $\sum_{\sigma} \sum_{j=1}^{r_{\sigma}} \frac{\beta_{\sigma,j} - [o(\sigma)-1]/2}{o(\sigma)(z(Q) - \lambda_{\sigma,j})}$ , and for arbitrary  $\chi$  the value of  $\beta_{\sigma,j}$  has to be replaced by  $\beta_{\sigma,j} + u_{\chi,\sigma}$  if that number is smaller than  $o(\sigma)$  and by  $\beta_{\sigma,j} + u_{\chi,\sigma} - o(\sigma)$  otherwise.

Now, since for a fixed  $\sigma$  the values of  $\chi(\sigma)$  are evenly distributed for  $\chi \in \hat{A}$  in the roots of unity of order  $o(\sigma)$ , the same assertion holds for the values of  $u_{\chi,\sigma}$  in the integers between 0 and  $o(\sigma) - 1$ , and the translation by  $\beta_{\sigma,j}$  (modified to remain in this set as above) does not alter the validity of this statement. Therefore for fixed  $\sigma$  and  $j$  and each  $0 \leq k < o(\sigma)$  the numerator over  $o(\sigma)(z(Q) - \lambda_{\sigma,j})$  equals  $k - \frac{o(\sigma)-1}{2}$  for precisely  $\frac{n}{o(\sigma)}$  values of  $\chi$ . The sum over  $\chi$  of these numbers is thus  $\frac{n}{o(\sigma)} \sum_{k=0}^{o(\sigma)-1} k - n \frac{o(\sigma)-1}{2}$ , an expression that is known to vanish by the formula for the sum of a finite arithmetic progression. Therefore the linear term in the expansion of  $F_e(P, Q) / \frac{\sqrt{dz(P)}\sqrt{dz(Q)}}{z(P)-z(Q)}$  vanishes, showing that

$$F_e(P, Q) = \frac{\sqrt{dz(P)}\sqrt{dz(Q)}}{z(P) - z(Q)} \left[ 1 + O\left((z(P) - z(Q))^2\right) \right]$$

around  $P = Q$ . Applying Theorem 3.4 and comparing the with the expansion from Proposition 3.1, we obtain  $\sum_{s=1}^g \frac{\partial \theta[e]}{\partial z_s} \Big|_{z=0} v_s(Q) = 0$ . As this equality holds for every  $Q$  and the differentials  $v_s$ ,  $1 \leq s \leq g$  are linearly independent on  $X$ , this completes the proof of the proposition.  $\square$

The product  $S[e](P, Q)S[-e](P, Q)$  will become useful below. For evaluating its expansion we shall need its expansion around the diagonal to the second order. For a divisor  $\Delta$  as above, two non-trivial elements  $\sigma$  and  $\rho$  from  $A$ , and two indices  $1 \leq j \leq r_\sigma$  and  $1 \leq i \leq r_\rho$ , we define

$$q_\Delta(\sigma, j; \rho, i) = \left( \frac{\beta_{\sigma, j}}{o(\sigma)} - \frac{o(\sigma) - 1}{2o(\sigma)} \right) \left( \frac{\beta_{\rho, i}}{o(\rho)} - \frac{o(\rho) - 1}{2o(\rho)} \right). \quad (3)$$

The value of  $q_\Delta(\sigma, j; \rho, i)$  for an arbitrary character  $\chi \in \hat{A}$  is therefore defined in the same manner, but with  $\beta_{\sigma, j}$  replaced by  $\beta_{\sigma, j} + u_{\chi, \sigma}$  in case that number does not exceed  $o(\sigma) - 1$  and by  $\beta_{\sigma, j} + u_{\chi, \sigma} - o(\sigma)$  if it does, and similarly for  $\rho$  and  $i$ . Finally, for one of the characteristics  $e$  appearing in Theorem 3.4 we set  $q_e(\sigma, j; \rho, i)$  to be defined as  $\sum_{\{\Delta | e = u(\Delta) + K\}} q_\Delta(\sigma, j; \rho, i)$ , which is a sum over  $n$  terms (an orbit of  $\hat{A}$ ). Using the proof of Proposition 3.5, we can now establish the following result, which also generalize statements from [Na], [Ko2], and [Ko3] to our general Abelian setting.

**Lemma 3.6.** *The expansion of the product  $S[e](P, Q)S[-e](P, Q)$  around the diagonal  $P = Q$  for  $Q$  neither a branch point nor a pole of  $z$  is*

$$\frac{dz(P)dz(Q)}{(z(P) - z(Q))^2} \left[ 1 + \frac{1}{n} \sum_{\sigma, j, \rho, i} \frac{q_e(\sigma, j; \rho, i)(z(P) - z(Q))^2}{(z(Q) - \lambda_{\sigma, j})(z(Q) - \lambda_{\rho, i})} + O((z(P) - z(Q))^3) \right].$$

*Proof.* We begin by evaluating the quadratic part of the term  $O[(z(P) - z(Q))^2]$  in the expansion of  $S[e](P, Q)$  around  $P = Q$  in Proposition 3.1. Theorem 3.4 allows us to consider the expansion of  $F_e(P, Q)$  instead, where we have seen in the proof of Proposition 3.5 that the summand associated with  $\chi$  is based on the expansion of  $\frac{f_{\chi\Delta}^+(P)}{\sqrt{dz(P)}}$  around  $P = Q$ , divided by  $\frac{f_{\chi\Delta}^+(Q)}{\sqrt{dz(Q)}}$ . While the linear term in the expansion of  $\frac{\varphi(z)}{\varphi(w)}$  for some function  $\varphi$  of  $z$  in the neighborhood of  $w$  concerns the derivative of  $\ln \varphi$ , for the second derivative we have to use  $\frac{d^2 \ln \varphi}{dz^2} + \left( \frac{d \ln \varphi}{dz} \right)^2$  evaluated at  $z = w$  (this is easily seen when one considers the second derivative of  $\ln \varphi$ ). For our function  $\frac{f_{\chi\Delta}^+}{\sqrt{dz}}$  the first derivative was evaluated in the proof of Proposition 3.5 to involve sums of a constant depending on  $\Delta$ ,  $\sigma$ ,  $j$ , and  $\chi$  divided by  $z - \lambda_{\sigma, j}$ , and the sum over  $\chi$  of these constants was seen there to vanish. The second derivative would be minus the same constant divided by  $(z - \lambda_{\sigma, j})^2$ , so that the sum over  $\chi$  vanishes again by the same argument. The square of the derivative here is now easily seen to be  $\sum_{\sigma, j, \rho, i} \frac{q_{\chi\Delta}(\sigma, j; \rho, i)}{(z(Q) - \lambda_{\sigma, j})(z(Q) - \lambda_{\rho, i})}$ , and taking the sum over  $\chi$  replaces the index  $\chi\Delta$  by  $e$  by definition. Recalling that the second derivative is the coefficient of  $(z(P) - z(Q))^2/2$  in the Taylor expansion and that we have a coefficient of  $\frac{1}{n}$ , the  $O[(z(P) - z(Q))^2]$  term in question is the latter sum multiplied by  $(z(P) - z(Q))^2/2n$ , plus  $O[(z(P) - z(Q))^3]$ .

We have to do the same for  $S[-e](P, Q)$ . But the proof of Lemma 3.3 shows that  $q_{\chi N\Delta}(\sigma, j; \rho, i) = q_{\chi\Delta}(\sigma, j; \rho, i)$  for every  $\Delta$ ,  $\chi$ ,  $\sigma$ ,  $j$ ,  $\rho$ , and  $i$  (since both multipliers in that expression are negated by that operation), thus showing that inverting the index  $e$  in  $q_{\chi\Delta}(\sigma, j; \rho, i)$  does not affect the result and  $S[-e](P, Q)$  has the same expansion as  $S[e](P, Q)$  around  $P = Q$  up to  $O[(z(P) - z(Q))^3]$  in the brackets. The required assertion thus follows from simple multiplication. This proves the lemma.  $\square$

In fact, the last stage in the proof of Lemma 3.6 could have been carried out using the equality  $S[e](P, Q) = S[-e](Q, P)$ , holding for general characteristics on Riemann surfaces, which is easily deduced from the properties of  $\theta$ . This equality also implies that the product considered in that lemma is a symmetric function of  $P$  and  $Q$ .

## 4 Constructing the Canonical Differential

The next object of which we shall make use is the *canonical differential* on  $X \times X$ , which is defined as follows.

**Proposition 4.1.** *There exists a unique meromorphic  $(1, 1)$ -form  $\omega$  on  $X \times X$  with the following properties: It is symmetric (i.e., satisfies  $\omega(P, Q) = \omega(Q, P)$  for every  $P$  and  $Q$  in  $X$ ); It is holomorphic when  $P \neq Q$  and has the singularity of the form*

$$\omega(P, Q) = \left( \frac{1}{(z(P) - z(Q))^2} + \frac{G_B(z(Q))}{6} + O(z(P) - z(Q)) \right) dz(P) dz(Q)$$

*along the diagonal  $P = Q$  in every coordinate  $z$  around a diagonal point; And its integrals (in  $P$  say, for fixed  $Q$ ) along any cycle  $a_i$ ,  $1 \leq i \leq g$  vanishes.*

Here  $G_B$  is the (holomorphic) *Bergman projective connection*, whose dependence on the the coordinate around  $Q$  is based on the fact that the difference between  $G_B(z(Q))dz(P)dz(Q)$  and the same expression with a different coordinate includes the Schwarzian derivative with respect to the coordinate change in question (see, e.g., the discussion around Equation (27) of [Fa]). Of course, replacing  $z(Q)$  by  $z(P)$  in the argument of  $G_B$  does not affect the formula, as the difference enters the error term  $O(z(P) - z(Q))$ . It also follows immediately from the symmetry in Proposition 4.1 that the  $a_i$ -integrals of  $\omega(P, Q)$  in  $Q$  (for fixed  $P$ ) vanish as well. As with  $E(P, Q)$  and  $S[e](P, Q)$ , the form of the singularity of  $\omega(P, Q)$  along the diagonal is invariant under coordinate changes.

Back in the case where  $X$  is an Abelian cover of  $\mathbb{CP}^1$  via the map  $z$  with the Abelian Galois group  $A$  of order  $n$ , what we now do is essentially decompose  $\omega$  according to the action of  $A$  on the two variables. More precisely, Proposition 1.2 has the following immediate corollary.

**Corollary 4.2.** *The space of holomorphic  $(1, 1)$ -forms on  $X \times X$  consists precisely of the sums over non-trivial  $\chi$  and  $\eta$  from  $\hat{A}$  of expressions of the form*

$q_\chi^\eta(z(P), z(Q))\psi_{\bar{\chi}}(P)\psi_\eta(Q)$ , where the degree of  $q_\chi^\eta$  in the first (resp. second) variable does not exceed  $t_\chi - 2$  (resp.  $t_\eta - 2$ ).

In particular, the degree bound in Corollary 4.2 implies that only characters  $\chi$  and  $\eta$  with  $t_\chi \geq 2$  and  $t_\eta \geq 2$  can appear (as is the case in Proposition 1.2 as well). The normalization with the subscript  $\chi$  corresponding to  $\psi_{\bar{\chi}}(P)$  will turn out more convenient below. While  $\omega$  is not holomorphic, we will establish a similar decomposition for it, taking its pole into consideration.

In order to do this, we shall need to verify the existence of certain polynomials. While this was relatively simple in the particular cases considered in [Na], [Ko2], [Ko3] and others, in our generality we shall have to use the following lemma.

**Lemma 4.3.** *Let two degrees  $d \geq 1$  and  $e \geq 1$  be given, and take two polynomials  $f_0$  and  $f_1$  in one variable, of degrees  $e+d$  and  $e+d-1$  respectively. A necessary and sufficient condition for the existence of polynomials  $f_l$ ,  $2 \leq l \leq d$ , each of degree  $d+e-l$ , such that the degree of  $\sum_{l=0}^d f_l(w)(z-w)^l$  in  $w$  will not exceed  $e$ , is that the leading coefficient of  $f_1$  is  $d$  times the leading coefficient of  $f_0$ .*

*Proof.* Write  $f_l(w) = \sum_{k=0}^{d+e-l} a_{lk}(-w)^k$ . Expanding  $(z-w)^l$  via the Binomial Theorem writes  $\sum_{l=0}^d f_l(w)(z-w)^l$  as  $\sum_{l=0}^d f_l(w) = \sum_{k=0}^{d+e-l} a_{lk}(-w)^k$ . Expanding  $(z-w)^l$  via the Binomial Theorem writes  $\sum_{l=0}^d f_l(w)(z-w)^l$  as  $\sum_{l=0}^d \sum_{k=0}^{d+e-l} \sum_{i=0}^l a_{lk} \binom{l}{i} z^i (-w)^{l+k-i}$ , which we can write using the variable  $j = k + l - i$  (so that  $k = j - l + i$ ) as  $\sum_{i=0}^d \sum_{l=i}^d \sum_{j=l-i}^{d+e-i} a_{l,j-l+i} \binom{l}{i} z^i (-w)^j$ . The required bound of the degree in  $w$  is satisfied if and only if each coefficient with  $j > e$  vanishes, which after interchanging the sums over  $l$  and  $j$  is seen to be the vanishing of  $\sum_{l=i}^{\min\{d,i+j\}} \binom{l}{i} a_{l,j-l+i}$  for every  $0 \leq i \leq d$  and  $j > e$ . As the sum of the indices of each coefficient appearing in the last sum is  $i+j$ , we may consider the equations arising from  $i$  and  $j$  separately according to their sum. We know that  $e < i+j \leq e+d$ , so that we fix the sum to be  $d+e-h$  for some  $0 \leq h < d$ , and obtain for each such  $h$  the equations  $\sum_{l=i}^{\min\{d,d+e-h\}} \binom{l}{i} a_{l,d+e-h-l}$  for  $0 \leq i < d-h$  (so that  $j = d+e-h-i > e$ ). There are  $d-h$  such linear equations, involving  $\min\{d+1, d+e-h+1\}$  indeterminates.

We begin with the maximal sum  $d+e$  (i.e., with  $h=0$ ), where we consider the  $d$  linear equations as equations in the coefficients  $a_{l,d+e-l}$  with  $1 \leq l \leq d$  with  $a_{0,d+e}$  viewed as a parameter. The resulting matrix  $M$ , in which the  $i$ th line corresponds to the equation with index  $i-1$ , has the  $il$ -entry  $\binom{l}{i-1}$ . The recursive relation for binomial coefficients  $\binom{r}{s} = \binom{r-1}{s} + \binom{r-1}{s-1}$  shows that  $M$  is obtained from the matrix  $T$  with entries  $\binom{l-1}{i-1}$  via left multiplication by the maximal lower Jordan matrix  $J$  with eigenvalue 1 (i.e., the matrix having 1s on the main diagonal and the one below it, and 0 anywhere else). As both matrices are triangular, they are invertible (hence so is  $M$ ), yielding a unique solution for the  $a_{l,d+e-l}$ s in terms of  $a_{0,d+e}$ . Turning to the equations arising from  $h > 0$ , we obtain linear equations represented by the matrix consisting of the first  $d-h$  rows of  $M$ . It therefore follows that by fixing the coefficients  $a_{l,d+e-l-h}$  with  $0 \leq l \leq h$ , the other ones are determined uniquely by them.

We recall that in the assumptions of the lemma, the coefficients with  $l = 0$  and with  $l = 1$  (i.e., those of  $f_0$  and  $f_1$ ) are given, and we have the freedom only in choosing the ones with  $l \geq 2$ . From the last paragraph it follows that once  $a_{1,d+e-1}$  attains the correct value according to  $a_{0,d+e}$ , it is possible to construct the other polynomials  $f_l$  with  $2 \leq l \leq d$  such that the degree bound in  $w$  is satisfied. In fact, we have many degrees of freedom in this construction (once the relation between the leading coefficients hold): The coefficients  $a_{l,d+e-h-l}$  with  $2 \leq l \leq h$  for any  $2 \leq h < d$  and all the coefficient  $a_{lk}$  with  $0 \leq k \leq e-l$  can be chosen freely. As for the required relation, the previous paragraph shows that  $a_{1,d+e-1}$  is obtained as the upper entry of the matrix  $M^{-1}$  multiplied by the vector having  $-a_{0,d+e}$  as its first entry and 0 in the rest, so that we have to evaluate the upper leftmost entry of  $M^{-1}$ . We have seen that  $M = JT$ , so that  $M^{-1} = T^{-1}J^{-1}$  and the entry that we seek is the product of the row sending the  $l$ th column of  $T$  to 1 if  $l = 1$  and 0 otherwise with the leftmost column of  $J^{-1}$ . The expansion of  $(1 - 1)^{l-1}$  shows that the row we seek has switching entries 1 and  $-1$ , and an easy calculation evaluates  $J^{-1}$  as the matrix whose  $k$ th lower diagonal is constantly  $(-1)^k$ . The required entry of  $M^{-1}$  is therefore  $d$ , so that the required relation is  $a_{1,d+e-1} = -da_{0,d+e}$ , and since we have expanded  $f_l$  in powers of  $-w$ , the leading coefficients are multiplied by opposite signs and the relation is indeed the required one. This proves the lemma.  $\square$

Note that the assumption  $\deg f_l \leq d + e - l$  is equivalent to the total degree of the sum in question will not exceed  $d + e$ .

We shall now construct a meromorphic  $(1, 1)$ -form on  $X \times X$  having the same singularities as  $\omega$ . Recall from Lemma 1.3 that  $y_\chi y_{\bar{\chi}}$  is a polynomial of degree  $t_\chi + t_{\bar{\chi}}$  in  $z$ , having precisely the numbers  $\lambda_{\sigma,j}$  with  $\sigma \in A$  such that  $\chi(\sigma) \neq 1$  and  $1 \leq j \leq r_\sigma$  as (simple) roots. Recall that the derivative of the function  $\ln y_\chi$  with respect to  $z$  is well-defined outside the roots of  $y_\chi y_{\bar{\chi}}$ . Indeed, raising  $y_\chi$  to the power  $n$  (or the exponent  $m$  of  $G$ , and even the order of  $\chi$  will suffice) we obtain a function in  $\mathbb{C}(X)$  that is a polynomial in  $z$  with the same roots as  $y_\chi y_{\bar{\chi}}$ , an expression whose derivative is indeed well-defined. We can therefore construct the following polynomial.

**Corollary 4.4.** *For every non-trivial character  $\chi \in \hat{A}$  there exists a polynomial  $\tilde{p}_\chi^\chi$  in two variables having the expansion  $\tilde{p}_\chi^\chi(z, w) = \sum_{l=0}^{t_\chi} \tilde{f}_{\chi,l}^\chi(w)(z - w)^l$  such that  $\tilde{f}_{\chi,0}^\chi = y_\chi y_{\bar{\chi}}$ ,  $\tilde{f}_{\chi,1}^\chi(z)$  is the product  $\tilde{f}_{\chi,0}^\chi(z) \sum_\sigma \frac{u_{\chi,\sigma}}{o(\sigma)} \sum_{j=1}^{r_\sigma} \frac{1}{z - \lambda_{\sigma,j}}$ , and the degree of  $\tilde{p}_\chi^\chi(z, w)$  in  $w$  does not exceed  $t_{\bar{\chi}}$ .*

*Proof.* Recalling that  $\deg \tilde{f}_{\chi,0}^\chi = t_\chi + t_{\bar{\chi}}$ , we first observe that  $\tilde{f}_{\chi,1}^\chi$  is a polynomial of degree one less. This is so since wherever  $u_{\chi,\sigma} \neq 0$  the numbers  $\lambda_{\sigma,j}$  are roots of  $\tilde{f}_{\chi,0}^\chi$ , and therefore the corresponding quotient remains a polynomial. The leading coefficients of  $\tilde{f}_{\chi,1}^\chi$  can now be seen as that of  $\tilde{f}_{\chi,0}^\chi$  multiplied by  $\sum_\sigma \frac{r_\sigma u_{\chi,\sigma}}{o(\sigma)}$ , which equals  $t_\chi$  by definition. Therefore the conditions of Lemma 4.3 are satisfied, yielding the existence of a polynomial  $\tilde{p}_\chi^\chi$  with the desired properties. This proves the corollary.  $\square$

Recall from the proof of Lemma 4.3 that in general we have some degree of freedom in defining  $\tilde{p}_\chi^\chi$ . Later we shall fix the choice (or more precisely modify  $\tilde{p}_\chi^\chi$  to a uniquely defined polynomial). As for the trivial character, the constant polynomial  $y_1^2$  satisfies all the conditions from Corollary 4.4, so that we take it to be  $\tilde{p}_1^1$ .

We now define, for every  $\chi \in \hat{A}$ , the  $(1, 1)$ -form

$$\xi_\chi(P, Q) = \frac{\tilde{p}_\chi^\chi(z(P), z(Q))}{(z(P) - z(Q))^2} \psi_{\bar{\chi}}(P) \psi_\chi(Q) = \frac{\tilde{p}_\chi^\chi(z(P), z(Q)) dz(P) dz(Q)}{y_\chi(P) s_{\bar{\chi}}(Q) (z(P) - z(Q))^2}$$

(so that in particular  $\xi_1(P, Q) = \frac{dz(P) dz(Q)}{(z(P) - z(Q))^2}$ ), and we let  $\xi$  be the average  $\frac{1}{n} \sum_{\chi \in \hat{A}} \xi_\chi$ . Note that here we avoid the need to normalize the  $y_\chi$ s since we defined  $\tilde{f}_{\chi,0}^\chi$  as the (not necessarily monic) product  $y_\chi y_{\bar{\chi}}$ , and not just the monic polynomial  $\prod_{\{\sigma \in A | \chi(\sigma) \neq 1\}} \prod_{j=1}^{r_\sigma} (z - \lambda_{\sigma,j})$ .

**Proposition 4.5.** *The  $(1, 1)$ -form  $\xi$  is holomorphic outside the diagonal in  $X \times X$ , and when  $Q$  is a point on  $X$  that is not a branch point of  $z$  and whose  $z$ -image is finite and  $P$  is in the neighborhood of  $Q$  then the expansion of  $\xi$  in  $z(P)$  around  $z(Q)$  is  $(\frac{1}{(z(P) - z(Q))^2} + O(1)) dz(P) dz(Q)$ .*

*Proof.* It is clear that if  $z(P)$  and  $z(Q)$  are distinct finite complex numbers then  $\xi$  is holomorphic at  $(P, Q)$  (since all the  $\xi_\chi$ s are). Assuming now that  $z(Q)$  is finite but  $z(P) = \infty$ , the order  $t_\chi - 2$  of  $\psi_\chi$  at  $P$  combines with the order 2 from the denominator to cancel the pole arising from  $\tilde{p}_\chi^\chi$ , since the degree of that polynomial in  $z$  cannot exceed  $t_\chi$  by construction. Inverting the roles of  $P$  and  $Q$ , the bound  $t_{\bar{\chi}}$  for the degree of  $\tilde{p}_\chi^\chi(z, w)$  in  $w$  yields the required holomorphicity by the same argument.

We now consider a point  $Q$  on which  $z$  is finite and not branched, fix  $\sigma \in A$ , and take  $P$  in the neighborhood of  $\sigma Q$ . Then we may expand  $y_\chi(P)$ , as in the proof of Proposition 3.5, as  $y_\chi(\sigma Q) + \frac{dy_\chi}{dP} \big|_{P=\sigma Q} (z(P) - z(Q))$  up to an error term of  $O[(z(P) - z(Q))^2]$ , so that the coefficient of  $dz(P) dz(Q)$  in  $\xi_\chi(P, Q)$  becomes

$$\frac{\tilde{f}_{\chi,0}^\chi(z(Q)) [1 + \sum_\sigma \frac{u_{\chi,\sigma}}{\sigma(\sigma)} \sum_{j=1}^{r_\sigma} \frac{1}{z - \lambda_{\sigma,j}} (z(P) - z(Q))]}{y_\chi(\sigma Q) y_{\bar{\chi}}(Q) [1 + \frac{d \ln y_\chi}{dP} \big|_{P=\sigma Q} (z(P) - z(Q))]} (z(P) - z(Q))^2 + O(1)$$

(since  $z(\sigma Q) = z(Q)$ ). Here all the terms with  $O[(z(P) - z(Q))^2]$  in both brackets go into the final  $O(1)$ . But since  $Q$  (hence also  $\sigma Q$ ) is not a branch point, the divisor of the function  $y_\chi$  shows that it can be presented as some constant multiple of  $\prod_\sigma \prod_{j=1}^{r_\sigma} (z - \lambda_{\sigma,j})^{u_{\chi,\sigma}/\sigma(\sigma)}$  (with the choice of fractional roots only affecting the external coefficient). Now, the derivative of the logarithm of this expression coincides (independently of the multiplying scalar) with the expression appearing in numerator, and the fact that  $y_\chi(\sigma Q) = \chi(\sigma) y_\chi(Q)$  and combines with the definition of  $\tilde{f}_{\chi,0}^\chi$  to conclude that  $\xi_\chi(P, Q)$  expands around

$P = \sigma Q$  as  $\left[\frac{\overline{\chi}(\sigma)}{(z(P)-z(Q))^2} + O(1)\right] dz(P)dz(Q)$ . The same orthogonality argument from the proof of Proposition 3.5, now shows that  $\xi = \frac{1}{n} \sum_{\chi \in \hat{A}} \xi_\chi$  is holomorphic at  $P = \sigma Q$  when  $\sigma$  is non-trivial and has the required singularity around  $P = Q$ . Once more the isolation of  $(\sigma Q, Q)$  for non-trivial  $\sigma$  and  $Q$  a branch point or a pole of  $z$  implies the holomorphicity of  $\xi$  also at these points, which completes the proof of the proposition.  $\square$

We immediately deduce the following corollary.

**Corollary 4.6.** *The  $(1,1)$ -form  $\omega - \xi$  is holomorphic on  $X \times X$ .*

*Proof.* Propositions 4.1 and 4.5 show that the poles of  $\omega$  and  $\xi$  coincide, except perhaps at the points  $(Q, Q)$  when  $Q$  is a branch point of  $z$  or when  $z(Q) = \infty$ . Since these are isolated points, the difference is holomorphic also at these points. This proves the corollary.  $\square$

In fact, the invariance of the singularity from Proposition 4.1 would have allowed us to extend the assertion about the singularity of  $\xi$  also to the points appearing in the proof of Corollary 4.6, relieving us from the need to consider them in the proof.

Corollary 4.2 now allows us to write the difference  $\omega(P, Q) - \xi(P, Q)$  as  $\frac{1}{n} \sum_{\chi, \eta} \widehat{p}_\chi^\eta(z(P), z(Q)) \psi_{\overline{\chi}}(P) \psi_\eta(Q)$  (only on non-trivial  $\chi$  and  $\eta$ ), with  $\widehat{p}_\chi^\eta$  polynomials with the aforementioned degree bounds. We extend the sum to go over all  $\chi$  and  $\eta$  by defining  $\widehat{p}_\chi^\eta$  to be 0 if  $\chi$  or  $\eta$  are trivial. An argument similar to Lemma 4.3 allows us to write  $\widehat{p}_\chi^\eta(z, w)$  as  $\sum_{l=2}^{t_\chi} \widehat{f}_{\chi, l}^\eta(w) (z - w)^{l-2}$ , where the degree of  $\widehat{f}_{\chi, l}^\eta$  not exceeding  $t_\chi + t_{\overline{\eta}} - l - 2$  (the reason for this choice of index will become apparent in Equation (4) below), and in particular  $\widehat{f}_{\chi, l}^\eta = 0$  for  $l \geq 2$  wherever  $\chi$  or  $\eta$  equal  $\mathbf{1}$ . Setting  $\widetilde{f}_{\chi, l}^\eta$  to be  $\widehat{f}_{\chi, l}^\eta + \widehat{f}_{\chi, l}^\eta$  if  $\eta = \chi \neq \mathbf{1}$  and  $l \geq 2$  and just  $\widehat{f}_{\chi, l}^\eta$  in case  $l \leq 1$  or  $\chi \neq \eta$ , so that  $p_\chi^\eta(z, w)$  is  $\widehat{p}_\chi^\eta(z, w) + \widehat{p}_\chi^\eta(z, w)(z - w)^2$  for equal characters and just  $\widehat{p}_\chi^\eta(z, w)$  otherwise, we find that

$$\omega(P, Q) = \sum_{\chi} \frac{p_\chi^\eta(z(P), z(Q)) dz(P) dz(Q)}{ny_\chi(P) y_{\overline{\chi}}(Q) (z(P) - z(Q))^2} + \sum_{\chi \neq \eta} \frac{p_\chi^\eta(z(P), z(Q)) dz(P) dz(Q)}{ny_\chi(P) y_{\overline{\eta}}(Q)}.$$

We remark that the fact that the singular part is based only on  $\chi = \eta$  is not coincidental, and corresponds to the fact that the diagonal action of  $G$  on  $X \times X$  has to preserve the form of the singularity. In fact, for later applications it will be more convenient to decompose the  $p_\chi^\eta$ s in terms of the  $p_{\chi, l}^\eta$ s. Recalling that both expressions  $\frac{p_\chi^\eta(z, w)}{(z - w)^2}$  and  $p_\chi^\eta(z, w)$  with  $\chi \neq \eta$  expand as  $\sum_{l=0}^{t_\chi} f_{\chi, l}^\eta(w) (z - w)^{l-2}$  (with the polynomials  $f_{\chi, 0}^\eta$  and  $f_{\chi, 1}^\eta$  being defined to be 0 if  $\chi \neq \eta$ ), we can write

$$\omega = \sum_{\eta \in \hat{A}} \frac{\omega_\eta}{n} \quad \text{with} \quad \omega_\eta(P, Q) = \sum_{\chi \in \hat{A}} \sum_{l=0}^{t_\chi} \frac{f_{\chi, l}^\eta(z(Q)) (z(P) - z(Q))^{l-2} dz(P) dz(Q)}{y_\chi(P) y_{\overline{\eta}}(Q)}. \quad (4)$$

We shall deduce some relations from the vanishing of  $\int_{P \in a_i} \omega(P, Q)$  for every  $Q \in X$  and  $1 \leq i \leq g$ . The calculations below will be simplified if we know that a similar assertion holds also for the  $(1, 1)$ -forms  $\omega_\eta$  with  $\eta \in \hat{A}$ .

**Lemma 4.7.** *For any  $\eta \in \hat{A}$  and any  $P$  and  $Q$  from  $X$  the expression  $\omega_\eta(P, Q)$  from Equation (4) equals  $\sum_{\sigma \in A} \overline{\eta(\sigma)} \omega(P, \sigma Q)$ . Moreover, for every such  $\eta$  and every  $1 \leq i \leq g$  we have  $\int_{P \in a_i} \omega_\eta(P, Q) = 0$ .*

*Proof.* The numerator of each summand in the definition of  $\omega_\eta$  depends on  $Q$  only through  $z(Q)$ , and is therefore invariant under replacing  $Q$  by  $\sigma Q$ . On the other hand, each denominator is multiplied by the same constant  $\overline{\eta(\sigma)}$ , so that  $\omega_\eta(P, \sigma Q) = \eta(\sigma) \omega_\eta(P, Q)$ . For any  $\chi \in \hat{A}$  the sum  $\sum_{\sigma \in A} \overline{\chi(\sigma)} \omega(P, \sigma Q)$  thus decomposes as  $\sum_{\sigma \in A} \sum_{\eta \in \hat{A}} \frac{\overline{\chi(\sigma)}}{n} \omega_\eta(P, \sigma Q)$ , and the latter multiplier was seen to equal  $\eta(\sigma) \omega_\eta(P, Q)$ . The only dependence on  $\sigma$  being now in the characters, we deduce from the orthogonality relations that the sum over  $\sigma$  is 1 if  $\chi = \eta$  and 0 otherwise. This establishes the first assertion, and the second one follows from the vanishing of the integral of each of the summands along each  $a_i$ . This proves the lemma.  $\square$

For the final analysis we shall also be needing an expression for the Bergman projective connection  $G_B$  from Proposition 4.1.

**Lemma 4.8.** *Around a point  $Q$  whose  $z$ -value is finite and not equal any  $\lambda_{\sigma,j}$  the value of  $G_B(z(Q))/6$  is*

$$\sum_{\chi \in \hat{A}} \sum_{\eta \in \hat{A}} \frac{f_{\chi,2}^\eta(z(Q))}{ny_\chi(Q)y_{\overline{\eta}}(Q)} - \frac{1}{2} \sum_{\sigma \in A} \sum_{\rho \in A} \sum_{j=1}^{r_\sigma} \sum_{i=1}^{r_\rho} \frac{\gamma_{\sigma,\rho} - \delta_{\sigma,\rho} \delta_{i,j} \frac{o(\sigma)-1}{2o(\sigma)}}{(z(Q) - \lambda_{\sigma,j})(z(Q) - \lambda_{\rho,i})}$$

times  $dz(Q)^2$ , where  $\gamma_{\sigma,\rho}$  is defined by the sum  $\frac{1}{n} \sum_{\chi \in \hat{A}} \frac{u_{\chi,\sigma} u_{\chi,\rho}}{o(\sigma)o(\rho)}$ , and  $\delta_{\sigma,\rho}$  and  $\delta_{i,j}$  Kronecker  $\delta$ -symbols (1 if the two indices coincide and 0 otherwise).

*Proof.* As in the proof of Lemma 3.6, we shall use the evaluations from Proposition 4.5 to a higher power of  $z(P) - z(Q)$ , recalling that we work with the formula for  $\omega$  rather than  $\xi$  from that proposition. We begin by expressing  $\omega_\eta(P, Q)$  for  $P$  around  $Q$ , up to an error term of  $z(P) - z(Q)$ . In the formula from Equation (4) all the terms with  $l \geq 3$  go into the error term, and we recall that  $f_{\chi,l}^\eta = 0$  for  $\chi \neq \eta$  if  $l$  is 0 or 1, so that the expression we consider is

$$\frac{f_{\eta,0}^\eta(z(Q))}{y_\eta(P)y_{\overline{\eta}}(Q)(z(P) - z(Q))^2} + \frac{f_{\eta,1}^\eta(z(Q))}{y_\eta(P)y_{\overline{\eta}}(Q)(z(P) - z(Q))} + \sum_{\chi \in \hat{A}} \frac{f_{\chi,2}^\eta(z(Q))}{y_\chi(P)y_{\overline{\eta}}(Q)}$$

times  $dz(P)dz(Q)$ . Putting the expression for  $y_\chi(P)$  from the proof of Proposition 4.5 in the denominator gives  $\frac{1}{y_\chi(Q)} \left[ 1 - \frac{d \ln y_\chi}{dP} \Big|_{P=Q} (z(P) - z(Q)) \right]$  up to



$O[(z(P) - z(Q))^2]$ , but we shall need the error term as well. An argument similar to the proof of Lemma 3.6 now shows that this error term is

$$\frac{1}{y_\chi(Q)} \left[ \left. \frac{d \ln y_\chi}{dP} \right|_{P=Q}^2 - \left. \frac{d^2 \ln y_\chi}{dP^2} \right|_{P=Q} \right] \frac{(z(P) - z(Q))^2}{2} + O((z(P) - z(Q))^3)$$

(recall that  $y_\chi$  is in the denominator, hence the inversion of the sign of its logarithm). Substituting, in the terms with  $l = 2$  we can just put  $P = Q$  inside the argument of  $y_\chi$ , and we recall that for  $l = 1$  (where  $\chi = \eta$ ) the polynomial  $f_{\eta,1}^\eta$  is  $f_{\eta,0}^\eta$  times the derivative of  $\ln y_\chi$ . This indeed yields the expansion  $\frac{1}{(z(P) - z(Q))^2} + O(1)$  from Proposition 4.5, where the  $O(1)$  is, up to  $O(z(P) - z(Q))$ , the expression

$$\sum_{\chi \in \hat{A}} \frac{f_{\chi,2}^\eta(z(Q))}{y_\chi(Q) y_\eta(Q)} - \frac{1}{2} \left[ \left. \frac{d \ln y_\eta}{dP} \right|_{P=Q}^2 + \left. \frac{d^2 \ln y_\eta}{dP^2} \right|_{P=Q} \right].$$

For the derivatives, we recall that  $\left. \frac{d \ln y_\eta}{dP} \right|_{P=Q}$  was  $\sum_\sigma \sum_{j=1}^{r_\sigma} \frac{u_{\eta,\sigma}/o(\sigma)}{z(Q) - \lambda_{\sigma,j}}$ , so that the second derivative is  $-\sum_\sigma \sum_{j=1}^{r_\sigma} \frac{u_{\eta,\sigma}/o(\sigma)}{(z(Q) - \lambda_{\sigma,j})^2}$ . Moreover, the latter expression is the contribution of  $\omega_\eta$ , so that in order to get  $G_B(z(Q))$  we also have to average over  $\eta$ . Expressions of the form  $\frac{1}{(z(Q) - \lambda_{\sigma,j})(z(Q) - \lambda_{\rho,i})}$  where  $(\sigma, j) \neq (\rho, i)$  (i.e., when the product of the  $\delta$ -symbols vanishes) get contributions only from the square, with the coefficient being  $\frac{1}{n} \sum_{\eta \in \hat{A}} \frac{u_{\eta,\sigma} u_{\eta,\rho}}{o(\sigma) o(\rho)}$ , which is  $\gamma_{\sigma,\rho}$  by definition (we do not have to multiply by 2, since in the required sum each such term is counted twice). As for the terms  $\frac{1}{(z(Q) - \lambda_{\sigma,j})^2}$ , here we have to consider the second derivative as well, which gives  $-\frac{u_{\eta,\sigma}}{o(\sigma)}$  to each  $\omega_\eta$ . Averaging the latter expressions over  $\eta$ , and recalling that for every number  $0 \leq u < o(\sigma)$  there exist precisely  $\frac{n}{o(\sigma)}$  characters  $\eta$  with  $u_{\eta,\sigma} = u$ , we get the sum  $-\frac{1}{o(\sigma)} \sum_{u=0}^{o(\sigma)-1} \frac{u}{o(\sigma)} = -\frac{o(\sigma)-1}{2o(\sigma)}$  (and the product of the  $\delta$  symbols indicate that this extra contribution appears only when  $\sigma = \rho$  and  $j = i$ ). This completes the proof of the lemma.  $\square$

## 5 Expanding Around Branch Points

Thomae's formula for our Abelian cover  $X$  of  $\mathbb{CP}^1$  will follow by connecting the objects defined in the previous sections, and comparing the coefficients in their expansions in a coordinate around a branch point. We begin by presenting the expressions involved in terms of the coordinate with which we shall work. If  $t$  is the (natural) local coordinate around a branch point lying over  $\lambda_{\sigma,j}$ , then it is related to  $z$  via the equality  $z - \lambda_{\sigma,j} = t^{o(\sigma)}$ .

**Lemma 5.1.** *Let  $e$  be one of the characteristics appearing in Theorem 3.4, and assume that  $Q$  is a point near a pre-image of  $\lambda_{\sigma,j}$ , for which we take the*

coordinate  $t$  from above around that branch point. For  $P$  in the neighborhood of  $Q$  (with the same coordinate  $t$ ) the product  $S[e](P, Q)S[-e](P, Q)$  expands as

$$1 + \frac{2o(\sigma)^2}{n} \sum_{(\rho, i) \neq (\sigma, j)} \frac{q_e(\sigma, j; \rho, i)}{\lambda_{\sigma, j} - \lambda_{\rho, i}} t(Q)^{o(\sigma)-2} (t(P) - t(Q))^2 + O((t(P) - t(Q))^3)$$

times  $\frac{dt(P)dt(Q)}{(t(P)-t(Q))^2}$ , plus an error term of  $O(t(Q)^{2o(\sigma)-2})$ .

*Proof.* We have to replace the coordinates for both  $P$  and  $Q$  in the expression from Lemma 3.6 from  $z$  to  $t$ , and begin with the analysis of  $\frac{dz(P)dz(Q)}{(z(P)-z(Q))^2}$ . From each differentials we get a multiplier of  $o(\sigma)$  times the appropriate  $t$  raised to the power  $o(\sigma) - 1$ , and if we expand the power of  $t(P)$  binomially then it becomes

$$t^{o(\sigma)-1} + (o(\sigma) - 1)t^{o(\sigma)-2}\Delta t + \frac{(o(\sigma) - 1)(o(\sigma) - 2)}{2}t^{o(\sigma)-3}(\Delta t)^2 + O((\Delta t)^3),$$

where  $\Delta t = t(P) - t(Q)$  and  $t$  stands for  $t(Q)$ . The denominator, which is now  $(t(P)^{o(\sigma)} - t(Q)^{o(\sigma)})^2$ , expands as  $(o(\sigma))^2(\Delta t)^2$  times the square of

$$t^{o(\sigma)-1} + \frac{(o(\sigma) - 1)}{2}t^{o(\sigma)-2}\Delta t + \frac{(o(\sigma) - 1)(o(\sigma) - 2)}{6}t^{o(\sigma)-3}(\Delta t)^2 + O((\Delta t)^3),$$

which after squaring and inverting becomes a multiplier of

$$t^{2-2o(\sigma)} - (o(\sigma) - 1)t^{1-2o(\sigma)}\Delta t - \frac{(o(\sigma) - 1)(7o(\sigma) - 11)}{12}t^{-2o(\sigma)}(\Delta t)^2 + O((\Delta t)^3).$$

By multiplying we find that this expression becomes  $\frac{dt(P)dt(Q)}{(\Delta t)^2}$  times an expression expanding as  $1 - \frac{(o(\sigma)-1)(o(\sigma)+1)}{12}\left(\frac{\Delta t}{t}\right)^2 + O((\Delta t)^3)$ .

Going over to the other multiplier from Lemma 3.6, we find that the error term  $O[(z(P) - z(Q))^3]$  is  $O((\Delta t)^3)$ , and from the expansion of  $(z(P) - z(Q))^2$  it suffices here to take just  $(o(\sigma))^2 t^{2o(\sigma)-2}(\Delta t)^2 + O((\Delta t)^3)$ . Now, each denominator  $z(Q) - \lambda_{\rho, i}$  becomes  $t^{o(\sigma)} + \lambda_{\sigma, j} - \lambda_{\rho, i}$ , which is just  $t^{o(\sigma)}$  in case  $\rho = \sigma$  and  $i = j$  and does not vanish in at  $t = 0$  otherwise, and we have products of two such denominators. Therefore all the summands in which neither of the branching values coincides with the value  $\lambda_{\sigma, j}$  goes into the error term  $O(t^{2o(\sigma)-2})$ . Observing now that the sum in Lemma 3.6 is symmetric under replacing  $\sigma$  by  $\rho$  and  $i$  by  $j$ , we may assume in the remaining terms that the first pair of indices are indeed  $\sigma$  and  $j$ , and then the terms in which  $(\rho, i) \neq (\sigma, j)$  are multiplied by 2 because of the contribution of the symmetric summand as well. Since in these terms the combination  $\frac{t^{o(\sigma)-2}}{t^{o(\sigma)} + \lambda_{\sigma, j} - \lambda_{\rho, i}}$  expands as  $\frac{t^{o(\sigma)-2}}{\lambda_{\sigma, j} - \lambda_{\rho, i}} + O(t^{2o(\sigma)-2})$ , this yields the required summands. But we also have the term in which  $\rho = \sigma$  and  $i = j$ , which equals  $\frac{o(\sigma)^2 q_e(\sigma, j; \sigma, j)(\Delta t)^2}{nt^2}$ . On the other hand, substituting  $\rho = \sigma$  and  $i = j$  in Equation (3) yields a square, and summing over  $\chi$  produces  $\frac{n}{o(\sigma)} \sum_{u=0}^{o(\sigma)-1} \left(\frac{u}{o(\sigma)} - \frac{o(\sigma)-1}{2o(\sigma)}\right)^2$  because of the distribution of the numbers  $u_{\chi, \sigma}$

for fixed  $\sigma$ . Multiplying by  $\frac{o(\sigma)^2}{n}$ , we obtain  $\frac{(o(\sigma)-1)(2o(\sigma)-1)}{6}$  from the sum of  $u^2$ ,  $-\frac{(o(\sigma)-1)^2}{2}$  from the sum over  $u$ , and  $+\frac{(o(\sigma)-1)^2}{4}$  from the constants, which combine to precisely  $\frac{(o(\sigma)-1)(o(\sigma)+1)}{12}$ . Hence multiplying the resulting expansion with the one arising from  $\frac{dz(P)dz(Q)}{(z(P)-z(Q))^2}$  yields the required result, as the two terms involving  $\left(\frac{\Delta t}{t}\right)^2$  cancel. This proves the lemma.  $\square$

**Lemma 5.2.** *Let  $Q$  and  $t$  be as in Lemma 5.1, but now take arbitrary  $P$  with  $z = z(P)$  (so that  $t = t(Q)$  only). Then the differential  $\omega_\eta(P, Q)$  from Equation (4) expands as*

$$(1 - \delta_{\eta(\sigma), 1}) \frac{o(\sigma)}{c_\eta} \left[ \frac{u_{\eta, \sigma}}{o(\sigma)} \frac{(f_{\eta, 0}^\eta)'(\lambda_{\sigma, j})}{y_\eta(P)(z - \lambda_{\sigma, j})} + \sum_{\chi \in \hat{A}} \sum_{l=0}^{t_\chi} \frac{f_{\chi, l}^\eta(\lambda_{\sigma, j})(z - \lambda_{\sigma, j})^{l-2}}{y_\chi(P)} \right]$$

times  $t^{o(\sigma)-1-u_{\overline{\eta}, \sigma}} dz dt$ , for some constant  $c_\eta \neq 0$ , up to an error term of  $O(t^{o(\sigma)-1})$ . In addition, the value of the Bergman projective connection in the coordinate  $t$  for  $Q$  expands as 6 times

$$\sum_{\chi \in \hat{A}} \sum_{\eta \in \hat{A}} \frac{o(\sigma)^2}{nc_{\overline{\chi}} c_\eta} f_{\chi, 2}^\eta(\lambda_{\sigma, j}) t^{2o(\sigma)-2-u_{\chi, \sigma}-u_{\overline{\eta}, \sigma}} - \sum_{(\rho, i) \neq (\sigma, j)} \frac{o(\sigma)^2 \gamma_{\sigma, \rho}}{\lambda_{\sigma, j} - \lambda_{\rho, i}} t^{o(\sigma)-2}$$

plus an error term of  $O(t^{o(\sigma)})$ .

*Proof.* Substituting  $z(Q) = \lambda_{\sigma, j} + t^{o(\sigma)}$  inside the polynomials  $f_{\chi, l}^\eta$  and the powers of  $z(P) - z(Q)$  allows us replace the value of  $z(Q)$  by  $\lambda_{\sigma, j}$  and obtain an error term of  $O(t^{o(\sigma)})$ . Moreover,  $dz(Q)$  is  $o(\sigma)t^{o(\sigma)-1}dt$ , and  $y_{\overline{\eta}}$  has order  $u_{\overline{\eta}, \sigma}$  at the branch point involved. More precisely, around the branch point in question,  $y_{\overline{\eta}}$  is some scalar multiple of  $t^{u_{\overline{\eta}, \sigma}}$  times a function that is the product of fractional powers of the terms  $z - \lambda_{\rho, i}$  for branching values different from  $\lambda_{\sigma, j}$  (with the choices of branches for these fractional powers going into the constant), so that it expands as  $t^{u_{\overline{\eta}, \sigma}}(c_\eta + O(t^{o(\sigma)}))$  for some non-zero constant  $c_\eta$ . Substituting all this into the defining expression of  $\omega_\eta$  in Equation (4) we obtain

$$\frac{o(\sigma)}{c_\eta} \sum_{\chi \in \hat{A}} \sum_{l=0}^{t_\chi} \left[ \frac{f_{\chi, l}^\eta(\lambda_{\sigma, j})(z - \lambda_{\sigma, j})^{l-2}}{y_\chi(P)} + O(t^{o(\sigma)}) \right] t^{o(\sigma)-1-u_{\overline{\eta}, \sigma}} dz dt,$$

with  $z$  and  $dz$  being those of  $P$ . Next, recall that only the character  $\chi = \eta$  contributes summands with  $l \leq 1$ , and we know the values of these polynomials. Now, if  $\eta(\sigma) = 1$  then the whole expression is  $O(t^{o(\sigma)-1})$ , which explains the multiplier  $1 - \delta_{\eta(\sigma), 1}$  and allows us to restrict attention to characters  $\eta$  with  $\eta(\sigma) \neq 1$ . But in this case the polynomial  $f_{\eta, 0}^\eta$  from Corollary 4.4 vanishes at  $\lambda_{\sigma, j}$ , allowing us omit this term. Moreover, the value of  $f_{\eta, 1}^\eta$  at  $\lambda_{\sigma, j}$  in this case is  $\frac{u_{\eta, \sigma}}{o(\sigma)} \cdot \frac{f_{\eta, 0}^\eta(w)}{w - \lambda_{\sigma, j}} \Big|_{w=\lambda_{\sigma, j}}$ , which coincides with  $\frac{u_{\eta, \sigma}}{o(\sigma)} (f_{\eta, 0}^\eta)'(\lambda_{\sigma, j})$  since the polynomial  $f_{\eta, 0}^\eta$  has distinct roots. This proves the first assertion.

For the second one, recall from, e.g., [Fa], that the difference between the expressions  $G_B(z(Q))dz(Q)^2$  and  $G_B(t)dt^2$  for a point  $Q$  in that neighborhood is the Schwarzian derivative of  $z$  with respect to  $t$ , defined by  $\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$  (where all the derivatives of  $z$  are with respect to  $t$ ). But as  $z$  is  $t^{o(\sigma)}$  plus a constant, the first quotient is  $\frac{(o(\sigma)-1)(o(\sigma)-2)}{t^2}$ , from which the subtraction of  $\frac{3}{2}\left(\frac{o(\sigma)-1}{t}\right)^2$  yields the value  $-\frac{(o(\sigma)-1)(o(\sigma)+1)}{2}$ . When we substitute the expressions for  $z(Q)$  and  $dz(Q)$  in terms of  $t$  and  $dt$  into the expression for  $G_B(z(Q))/6$  from Lemma 4.8, the proof of the first assertion here shows that the summand associated with  $\chi$  and  $\eta$  expands as  $\frac{o(\sigma)^2}{nc_{\chi,\eta}}f_{\chi,2}^\eta(\lambda_{\sigma,j})t^{2o(\sigma)-2-u_{\chi,\sigma}-u_{\eta,\sigma}} + O(t^{3o(\sigma)-2-u_{\chi,\sigma}-u_{\eta,\sigma}})$ . The fact that the  $u$ -coefficients are bounded by  $o(\sigma) - 1$  shows that all these  $O$  terms can be included in  $O(t^{o(\sigma)})$ . Now, the expansion of the denominators  $z - \lambda_{\rho,i}$  appearing in the proof of Lemma 5.1 shows that summands in which neither of the denominators involve the point  $\lambda_{\sigma,j}$  around whose pre-image we expand using  $t$  are all  $O(t^{2o(\sigma)-2})$ , and we can use again the symmetry argument since  $\gamma_{\sigma,\rho}$  is symmetric in its indices. Following the same proof, the terms in which  $(\rho,i) \neq (\sigma,j)$  contribute indeed the remaining terms (as here the product of the  $\delta$  factors vanishes, and we have the external factor  $\frac{1}{2}$ ), with another error estimate of  $O(t^{2o(\sigma)-2})$  (and the two latter error estimate can also be included in  $O(t^{o(\sigma)})$  since  $o(\sigma) \geq 2$  for a non-trivial element  $\sigma \in A$ ). Considering the last term, with  $\rho = \sigma$  and  $i = j$ , we obtain a contribution of  $\frac{o(\sigma)^2\gamma_{\sigma,\sigma}-o(\sigma)(o(\sigma)-1)/2}{2t^2}$ . But in the evaluation of  $o(\sigma)^2\gamma_{\sigma,\sigma} = \frac{1}{n}\sum_{\chi \in \hat{A}}u_{\chi,\sigma}^2$  we find that every number  $0 \leq u < o(\sigma)$  appears  $\frac{n}{o(\sigma)}$  times, yielding the value  $\frac{o(\sigma)-1}{6}(2o(\sigma)-1)$ . Subtracting  $\frac{o(\sigma)(o(\sigma)-1)}{2}$  and dividing by  $2t^2$  yields, as in the proof of Lemma 5.1, the value  $-\frac{(o(\sigma)-1)(o(\sigma)+1)}{12}$  of the Schwarzian derivative divided by 6. The fact that this term comes with a minus sign implies that it cancels with the Schwarzian derivative (indeed, the resulting expression has to be holomorphic at  $t = 0$  as well), which completes the proof of the lemma.  $\square$

The fact that the expression differentiating  $\frac{dz(P)dz(Q)}{(z(P)-z(Q))^2}$  from  $\frac{dt(P)dt(Q)}{(t(P)-t(Q))^2}$  in Lemma 5.1 is the same (in the appropriate order approximation) as the Schwarzian derivative appearing in the proof of the second assertion of Lemma 5.2 is not coincidental. In fact, a direct proof for the transformation of the Bergman projective connection with respect to coordinate changes on general Riemann surfaces (involving the general Schwarzian derivative) can be obtained by evaluating the numerator and denominator in  $\frac{dz(P)dz(Q)}{(z(P)-z(Q))^2}$  in terms of  $t$  up to  $(\Delta t)^3$  (like we did in the proof of Lemma 5.1), and taking the appropriate limit of the difference as  $P \rightarrow Q$ . We also remark that the terms with  $\chi$  and  $\eta$  such that  $u_{\chi,\sigma} + u_{\eta,\sigma} \leq o(\sigma) - 2$  can also be absorbed into the error term, though only a particular case of this fact will be of use to us.

It now becomes evident that the coefficients  $q_e(\sigma, j; \rho, i)$  and  $\gamma_{\sigma,\rho}$  will play a role in what follows. We would therefore like to obtain simplified expressions for them, from which we may draw more information about them. Consider now an integer  $d$ , and a class  $h \in (\mathbb{Z}/d\mathbb{Z})^\times$ . For such  $d$  and  $h$  we define the function

$\phi_{h+d\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{C}$  by

$$\phi_{h+d\mathbb{Z}}(s) = \sum_{0 \neq k \in \mathbb{Z}/d\mathbb{Z}} \frac{\mathbf{e}(ks/d)}{(1 - \mathbf{e}(kh/d))(1 - \mathbf{e}(-k/d))} \quad (5)$$

(which is essentially a function from  $\mathbb{Z}/d\mathbb{Z}$  to  $\mathbb{C}$ ). The value of  $\phi_{h+d\mathbb{Z}}$  is essentially a Dedekind sum (up to simple additive and multiplicative constants depending only on  $d$ ), so that one can refer to  $\phi_{h+d\mathbb{Z}}(s)$  as some sort of generalized Dedekind sum. Now, given two elements  $\sigma$  and  $\rho$  of  $A$ , set  $d = |\langle \sigma \rangle \cap \langle \rho \rangle|$ , a divisor of both  $o(\sigma)$  and  $o(\rho)$ . As the intersection group whose cardinality defines  $d$  is generated by either  $\sigma^{o(\sigma)/d}$  or  $\rho^{o(\rho)/d}$ , define  $h \in (\mathbb{Z}/d\mathbb{Z})^\times$  by the equality  $\rho^{o(\rho)/d} = (\sigma^{o(\sigma)/d})^h$ . The relation of these functions to the coefficients in question is as follows.

**Proposition 5.3.** *For a characteristic  $e$  from Theorem 3.4, take a divisor  $\Delta$  as in Theorem 1.4 with  $u(\Delta) + K = e$ , and write  $\Delta$  as in Equation (1). Then the value of the number  $q_e(\sigma, j; \rho, i)$  is  $\frac{n}{o(\sigma)o(\rho)} \cdot \phi_{h+d\mathbb{Z}}(\beta_{\rho, i} - h\beta_{\sigma, j})$ , while  $\gamma_{\sigma, \rho}$  equals  $\frac{\phi_{h+d\mathbb{Z}}(0)}{o(\sigma)o(\rho)} + \frac{(o(\sigma)-1)(o(\rho)-1)}{4o(\sigma)o(\rho)}$ .*

*Proof.* Recall that when replacing  $\Delta$  by  $\chi\Delta$  in the index of  $q_\Delta(\sigma, j; \rho, i)$ , we have to put the number  $\beta_{\sigma, j} + u_{\chi, \sigma} - o(\sigma)\delta(u_{\chi, \sigma} \geq o(\sigma) - \beta_{\sigma, j})$ , where the symbol with  $\delta$  equals 1 if the condition in brackets is satisfied and vanishes otherwise, in the place of  $\beta_{\sigma, j}$ . A similar expression has to appear instead of  $\beta_{\rho, i}$ . Shortening the notation of the  $\delta$  terms to just  $\delta_{\sigma, j, \chi}$  and  $\delta_{\rho, i, \chi}$ , we find that  $q_e(\sigma, j; \rho, i)$  is the sum of  $n$  terms, each of which is the product of four numbers, only two of which depends on  $\chi$ . The fact that the sum over  $\chi$  of one multiplier vanishes, already used in the proofs of Proposition 3.5 and Lemma 3.6, implies that  $q_e(\sigma, j; \rho, i)$  reduces to  $\sum_{\chi \in \hat{A}} (\frac{u_{\chi, \sigma}}{o(\sigma)} - \delta_{\sigma, j, \chi})(\frac{u_{\chi, \rho}}{o(\rho)} - \delta_{\rho, i, \chi}) - nq_\Delta(\sigma, j; \rho, i)$ . Noting that for  $\beta_{\sigma, j} = \beta_{\rho, i} = 0$  the two  $\delta$  terms vanish for all  $\chi$ , the sum over  $\chi$  in this case gives just  $n\gamma_{\sigma, \rho}$ , so that the same calculation evaluates both of the required coefficients.

Now, the group generated by  $\sigma$  and  $\rho$  has order  $\frac{o(\sigma)o(\rho)}{d}$ , so that for every pair of possible values for  $u_{\chi, \sigma}$  and  $u_{\chi, \rho}$ , this pair of values is attained by precisely  $\frac{nd}{o(\sigma)o(\rho)}$  characters from  $\hat{A}$ . Moreover, for a pair of numbers,  $0 \leq u < o(\sigma)$  and  $0 \leq v < o(\rho)$  say, to satisfy  $u = u_{\chi, \sigma}$  and  $v = u_{\chi, \rho}$  for the same character  $\chi \in \hat{A}$ , it is necessary that  $v \equiv hu \pmod{d}$ . Indeed, we have  $\chi(\rho^{o(\rho)/d}) = \mathbf{e}(u_{\chi, \rho}/d)$  and  $\chi((\sigma^{o(\sigma)/d})^h) = \mathbf{e}(hu_{\chi, \sigma}/d)$ , and the two arguments of  $\chi$  here coincide. Since this restriction leaves  $\frac{o(\sigma)o(\rho)}{d}$  pairs, this condition is also sufficient. We can therefore write the sum over  $\chi$  that we wish to evaluate as  $\frac{nd}{o(\sigma)o(\rho)}$  times  $\sum_{u=0}^{o(\sigma)-1} \sum_{v=0}^{o(\rho)-1} (\frac{u}{o(\sigma)} - \delta_{\sigma, j, u})(\frac{v}{o(\rho)} - \delta_{\rho, i, v})\delta(v \equiv hu \pmod{d})$ , where replacing the index  $\chi$  by  $u$  or  $v$  in the former  $\delta$  terms corresponds to replacing the condition on  $u_{\chi, \sigma}$  or  $u_{\chi, \rho}$  by  $u$  or  $v$  respectively, and the last  $\delta$  term is defined by a similar rule of 1 if the condition is true and 0 if it is false. But using the usual orthogonality of roots of unity, the latter  $\delta$  term can be replaced by  $\frac{1}{d} \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \mathbf{e}(k(hu - v)/d)$ . The external coefficient

thus reduces to  $\frac{n}{o(\sigma)o(\rho)}$ , and for every  $k \in \mathbb{Z}/d\mathbb{Z}$  we have the product of the sums  $\sum_{u=0}^{o(\sigma)-1} (\frac{u}{o(\sigma)} - \delta_{\sigma,j,u}) \mathbf{e}(khu/d)$  and  $\sum_{v=0}^{o(\rho)-1} (\frac{v}{o(\rho)} - \delta_{\rho,i,v}) \mathbf{e}(-kv/d)$ . We can also simplify the second term in these sums to  $\sum_{u=o(\sigma)-\beta_{\sigma,j}}^{o(\sigma)-1} \mathbf{e}(khu/d)$  and  $\sum_{v=o(\rho)-\beta_{\rho,i}}^{o(\rho)-1} \mathbf{e}(-kv/d)$  respectively.

Let us evaluate these sums, starting with the with  $k = 0$  and trivial roots of unity. Then  $\sum_{u=0}^{o(\sigma)-1} \frac{u}{o(\sigma)} = \frac{o(\sigma)-1}{2}$  and  $\sum_{v=0}^{o(\rho)-1} \frac{v}{o(\rho)} = \frac{o(\rho)-1}{2}$ , while the sums arising from  $\delta$  reduce to just  $\sum_{u=o(\sigma)-\beta_{\sigma,j}}^{o(\sigma)-1} 1 = \beta_{\sigma,j}$  and  $\sum_{v=o(\rho)-\beta_{\rho,i}}^{o(\rho)-1} 1 = \beta_{\rho,i}$ . Recalling the external coefficient  $\frac{n}{o(\sigma)o(\rho)}$ , the resulting expression cancels with the term  $nq_{\Delta}(\sigma, j; \rho, i)$ , while for  $\gamma_{\sigma,\rho}$ , in which the  $\beta$  numbers vanish (and we have an extra  $n$  in the denominator), it produces the desired term that does not contain  $\phi_{h+d\mathbb{Z}}$ . For a non-zero index  $k$ , the sums with  $\delta$  are sums of geometric progressions, yielding  $\frac{1-\mathbf{e}(-kh\beta_{\sigma,j}/d)}{1-\mathbf{e}(kh/d)}$  and  $\frac{1-\mathbf{e}(k\beta_{\rho,i}/d)}{1-\mathbf{e}(-k/d)}$  respectively. On the other hand, we have the equality  $\sum_{l=0}^{m-1} ly^{l-1} = \frac{1-y^m}{(1-y)^2} - \frac{my^{m-1}}{1-y}$  for any  $y \neq 1$  (see the proof of Proposition 6.5 of [KZ]). In our case the remaining sums over  $u$  and  $v$  involve the summand  $y^u$  and  $y^v$  for  $y$  satisfying  $y^m = 1$  (since  $d$  divides both  $o(\sigma)$  and  $o(\rho)$ ), so after multiplying the latter equality by  $y$  and substituting  $y^m = 1$  we find that these sums equal  $-\frac{1}{1-\mathbf{e}(kh/d)}$  and  $-\frac{1}{1-\mathbf{e}(-k/d)}$  respectively. Combining, we find that the summand associated with  $k$  is the product of  $\frac{-\mathbf{e}(-kh\beta_{\sigma,j}/d)}{1-\mathbf{e}(kh/d)}$  and  $\frac{-\mathbf{e}(k\beta_{\rho,i}/d)}{1-\mathbf{e}(-k/d)}$ , and after taking the sum over  $k$  we indeed obtain the definition of  $\phi_{h+d\mathbb{Z}}$  from Equation (5). After multiplying by the required external coefficients, this completes the proof of the proposition.  $\square$

Note that the argument of  $\phi_{h+d\mathbb{Z}}$  in Proposition 5.3, which seems to depend on the divisor  $\Delta$  representing  $e$ , is a function of  $e$  alone. Indeed, if we use  $\chi\Delta$  instead of  $\Delta$  the argument would have been  $(\beta_{\rho,i} + u_{\chi,\rho}) - h(\beta_{\sigma,j} + u_{\chi,\sigma})$  (up to multiples of  $o(\sigma)$  and of  $o(\rho)$ , hence of  $d$ ). But the proof of Proposition 5.3 has shown that for our  $\sigma$  and  $\rho$  the numbers  $u_{\chi,\rho}$  and  $u_{\chi,\sigma}$  satisfy the congruence  $u_{\chi,\rho} \equiv hu_{\chi,\sigma} \pmod{d}$ , so that the residue modulo  $d$  of that argument (which determines its value under  $\phi_{h+d\mathbb{Z}}$ ) coincides with  $\beta_{\rho,i} - h\beta_{\sigma,j}$ .

We shall obtain our main result by deducing relations between the various objects appearing above, and using our explicit formulae for them. The first relation is based on Corollary 2.12 of [Fa] (alluded to also in [Na], [Ko2], and [Ko3]), which we now quote. Recall that  $v_s$ ,  $1 \leq s \leq g$  are the basis for the differentials of the first kind on  $X$  that is dual to the canonical homology basis chosen for  $X$ .

**Proposition 5.4.** *Let  $X$  be a compact Riemann surface, and let  $e$  be a point  $e \in J(X)$  such that  $\theta[e](0, \tau) \neq 0$ . Then one has the equality*

$$S[e](P, Q)S[-e](P, Q) = \omega(P, Q) + \sum_{r=1}^g \sum_{s=1}^g \frac{\partial^2 \ln \theta[e]}{\partial z_r \partial z_s} \Big|_{z=0} v_r(P) v_s(Q).$$

We remark that while [Fa] writes this equality in terms of  $\theta$  around  $e$  rather than  $\theta[e]$  around 0, the fact that the latter is the product of the former times the exponential of a linear function of  $\{z_s\}_{s=1}^g$  (which are not related to the coordinate  $z$  taken on  $X$ ) shows that the second derivatives of their logarithms indeed coincide.

Back to our setting, we have explicit expressions for both sides of the equality from Proposition 5.4, from which we can obtain interesting relations. However, the equalities that we shall later require are consequences of Proposition 5.4 in the coordinate  $t$ , using the expressions from Lemmas 5.1 and 5.2. For this we recall that  $\frac{v_s}{dt}$  is a function of  $t$  around our branch point, which can be multiplied and differentiated. The equality which will be essential in proving Thomae is the following one.

**Corollary 5.5.** *For every characteristic  $e$  considered in Theorem 3.4 and every branching value  $\lambda_{\sigma,j}$  the expression*

$$n \sum_{(\rho,i) \neq (\sigma,j)} \frac{2\phi_{h+d\mathbb{Z}}(s) + \phi_{h+d\mathbb{Z}}(0) + \frac{(o(\sigma)-1)(o(\rho)-1)}{4}}{o(\sigma)o(\rho)(\lambda_{\sigma,j} - \lambda_{\rho,i})} - \sum_{\{\chi \in \hat{A} | \chi(\sigma) \neq 1\}} \frac{f_{\chi,2}^\chi(\lambda_{\sigma,j})}{(f_{\chi,0}^\chi)'(\lambda_{\sigma,j})},$$

where  $d$  and  $h$  arise from  $\sigma$  and  $\rho$  as in Proposition 5.3 and  $s$  is the class of  $\beta_{\rho,i} - h\beta_{\sigma,j}$  modulo  $d\mathbb{Z}$ , equals

$$\frac{1}{o(\sigma)(o(\sigma)-2)!} \sum_{r=1}^g \sum_{s=1}^g \left. \frac{\partial^2 \ln \theta[e]}{\partial z_r \partial z_s} \right|_{z=0} \sum_{Q \in f^{-1}(\lambda_{\sigma,j})} \frac{d^{o(\sigma)-2}}{dt^{o(\sigma)-2}} \left( \frac{v_r}{dt} \cdot \frac{v_s}{dt} \right) \Big|_{t=0},$$

where for each  $Q$  in the sum the symbol  $t$  represents the corresponding coordinate around that point  $Q$ .

*Proof.* Consider the sum  $\sum_{\rho \in A} \Omega(\rho P, \rho Q)$  where  $\Omega$  stands for each of the three terms appearing in Proposition 5.4, and assume that  $P$  is close to  $Q$ , using the coordinate  $t$ . The expansion of  $S[e](P, Q)S[-e](P, Q)$ , either in Lemma 3.6 or in Lemma 5.1, is invariant under letting the same element  $\rho \in A$  act on both  $P$  and  $Q$  (since both points appear only via their  $z$ -values in that expansion), so that the sum produces  $n$  times the expansion in  $t$  appearing in Lemma 5.1. The same assertion holds for the sums on  $\rho$  and  $i$  (as well as on  $\sigma$  and  $j$ ) in Lemmas 4.8 and 5.2. On the other hand, we recall that the constants  $c_{\overline{\chi}}$  and  $c_{\eta}$  appearing in Lemma 5.2 arise from the functions  $y_{\chi}$  and  $y_{\overline{\eta}}$  appearing in Lemma 4.8. Applying  $\rho$  to  $Q$  in these lemmas therefore multiplies the summand associated with  $\chi$  and  $\eta$  by  $\eta(\rho)\overline{\chi(\rho)}$ , and the summing thus leaves (by orthogonality) only the terms with  $\chi = \eta$ , canceling the  $n$  from the denominator. Now, the ones in which this common character contains  $\sigma$  in its kernel are absorbed in the  $O(t^{o(\sigma)})$  term (they are even  $O(t^{2o(\sigma)-2})$ ), while in the other ones, which multiply  $t^{o(\sigma)-2}$  because of Lemma 1.3, the denominator  $c_{\chi}c_{\overline{\chi}}$  is by definition the limit of  $\frac{y_{\chi}y_{\overline{\chi}}}{z-\lambda_{\sigma,j}}$  at any pre-image of  $\lambda_{\sigma,j}$ . As the numerator is the polynomial  $f_{\chi,0}^\chi$  from Corollary 4.4, which is known to have a simple zero at  $\lambda_{\sigma,j}$ , this quotient is indeed the asserted derivative.

Therefore the first asserted expression is the coefficient of  $t^{o(\sigma)-2}dt^2$  in the expansion of  $\frac{1}{o(\sigma)^2} \sum_{\rho \in A} \lim_{P \rightarrow Q} (S[e](\rho P, \rho Q)S[-e](\rho P, \rho Q) - \omega(\rho P, \rho Q))$  when  $P$  approaches  $Q$  and  $Q$  lies near a branch point (with coordinate  $t$ ), after one substitutes the value of  $q_e(\sigma, j; \rho, i) + n\gamma_{\sigma, \rho}$  given in Proposition 5.3. We must therefore show that the second asserted expression is the coefficient of that power of  $t$  (times  $dt^2$ ) in  $\frac{1}{o(\sigma)^2}$  times the sum over  $\rho$  of the expression resulting from the remaining term in Proposition 5.4 with  $P = Q$  in terms of  $t$ . Now, in the contribution arising from each pair of indices  $r$  and  $s$ , the derivatives of  $\ln \theta[e]$  are independent of the points, and we may substitute  $P = Q$  in the differentials  $v_r$ , and take the expansion in the variable  $t$  around the pre-image of  $\lambda_{\sigma, j}$ . The coefficient of  $t^{o(\sigma)-2}$  in the expansion of  $\frac{v_r}{dt} \cdot \frac{v_s}{dt}$  is evaluated using Taylor's Theorem, and after we take the sum over  $\rho$  we find that the expansion around each of the  $\frac{n}{o(\sigma)}$  points mapping to  $\lambda_{\sigma, j}$  arises from  $o(\sigma)$  different values of  $\rho$ . Around each such point, the action of  $\sigma$  multiplies  $t$  and  $dt$  by  $e(\frac{1}{o(\sigma)})$ , so that  $t^{o(\sigma)-2}dt^2$  is unaffected by this operation, and we indeed get  $o(\sigma)$  identical contributions from the elements of each such orbit. Canceling this multiplicity with the external coefficient  $\frac{1}{o(\sigma)^2}$ , we indeed obtain the desired expression. This completes the proof of the corollary.  $\square$

Using the Taylor expansion of every  $\frac{v_s}{dt}$  around  $t = 0$  as  $\sum_{\alpha=0}^{\infty} \frac{v_s^{(\alpha)}}{\alpha!}(Q)t^\alpha$  around such  $Q$ , the derivative in question becomes the familiar expression  $\sum_{\alpha=0}^{o(\sigma)-2} \binom{o(\sigma)-2}{\alpha} v_r^{(\alpha)}(Q)v_s^{(o(\sigma)-2-\alpha)}(Q)$  from [Na] and [Ko2] around such  $Q$  (this also applies to [Ko3], but there  $o(\sigma) = 2$  for every non-trivial  $\sigma$ , so that the sum reduces to a single term). However, the form appearing in Corollary 5.5 will be more useful for us later.

The proof of Corollary 5.5 in fact shows that doing the averaging process in the second part of Lemma 5.2 reduces the first sum there to only the terms with  $\chi = \eta$ , with the error term being improved to  $O(t^{2o(\sigma)-2})$  (since this becomes an  $A$ -expression expression, whence an expression in terms of  $z$  alone). It also implies that the expansion of each product  $v_r(Q)v_s(Q)$  around a branch point contains, after such an averaging, only powers of  $t$  that are congruent to  $-2$  modulo  $o(\sigma)$  (multiplied by  $dt^2$ ), in correspondence with  $\omega(P, Q)$  and the product from Lemma 3.6 taking the same form. Also note that this averaging process did not appear in [Na] and [Ko2], and these references go directly to compare the coefficients of  $t^{n-2}$ . The reason is that they consider the case where  $A$  is cyclic and  $\sigma$  is a generator, so that the value of  $u_{\chi, \sigma}$  determines  $\chi$  and the only possibility for  $u_{\chi, \sigma} + u_{\overline{\eta}, \sigma}$  to equal  $o(\sigma) = n$  (to yield a contribution to the coefficient of  $t^{n-2}$ ) is where  $\chi = \eta$  and  $\sigma$  is not in the kernel of that character. Moreover, the averaging in their case does not affect the terms with  $v_s$  and  $v_r$ , since in a fully ramified  $Z_n$  curve each branching value is the image of a single branch point. Since in our general case the required coefficient we still contain  $f_{\chi, 2}^\eta(\lambda_{\sigma, j})$  for  $\eta \neq \chi$  (this will always be the case when  $\chi(\sigma) = \eta(\sigma) \neq 1$ ), we need to average in order to leave only the elements we want. The reason for aiming to have only terms with  $\chi = \eta$ , which already appears in [Na] and [Ko2], will become apparent in Proposition 6.2.



## 6 The Thomae Formulae

The expressions from Corollary 5.5 can be interpreted in terms of variations of the complex structure on  $X$  with respect to the branching value  $\lambda_{\sigma,j}$  in the neighborhood of whose pre-images we work. Indeed, topologically  $X$  is a real compact surface of genus  $g$ , and  $a_i$  and  $b_i$  with  $1 \leq i \leq g$  form a basis for its integral homology which is canonical with respect to the intersection pairing. We shall choose explicit paths representing these homology classes, assume that none of them pass through any branch point or any pole of  $z$ , and we allow ourselves the usual abuse of notation in which  $a_i$  and  $b_i$  denotes these explicit paths as well as the homology classes they represent. This may be important in case the resulting integrals depends on the explicit choice of path.

A small perturbation in the value of a single  $\lambda_{\sigma,j}$  will not change the genus of  $X$  as long as  $\lambda_{\sigma,j}$  does not coincide with another branching value, and for small enough perturbations  $\lambda_{\sigma,j}$  will not land on the representing paths that we chose for the  $a_i$ s and the  $b_i$ s. We may therefore consider the real manifold underlying  $X$  and the choice of homology basis  $a_i$  and  $b_i$  with  $1 \leq i \leq g$  as fixed during such a perturbation. Hence the dependence of any integrals we encounter on  $\lambda_{\sigma,j}$  is only via its integrand, so that the derivatives of these integrals are evaluated by integrating the derivatives of the integrands along the same paths.

The first object that we analyze in this way is based on the differentials  $\{z^k \psi_\chi | 0 \leq k \leq t_\chi - 2\}$ , which forms a basis for the space of differentials of the first kind on  $X$  by Proposition 1.2. Combining them with the  $a_i$  part of the homology of  $X$ , we obtain a matrix  $C \in M_{g \times g}(\mathbb{C})$  with entries  $\int_{a_i} z^k \psi_\chi$  (where we assume that columns correspond to differentials and rows to homology elements). The matrix  $C$  is evidently invertible—it is the transition matrix between the basis  $v_s$ ,  $1 \leq s \leq g$  and the basis from Proposition 1.2, and its dependence on the value of  $\lambda_{\sigma,j}$  is clearly holomorphic (since so are the integrands  $z^k \psi_\chi$ ). Therefore so is the dependence of its determinant, the derivative of the logarithm of which we shall evaluate in the following lemma (again, generalizing similar statements from [Na], [Ko2], and [Ko3]). Note that the ordering of the differentials or of the  $a_i$ s is not important, since signs do not affect derivatives of logarithms.

**Lemma 6.1.** *Let  $B_{\sigma,j}$  denote the matrix defined just like  $C$  but with each occurrence of  $z$  replaced by  $z - \lambda_{\sigma,j}$ . For any  $\chi \in \hat{A}$  with  $t_\chi \geq 2$  (in particular  $\chi \neq \mathbf{1}$ ) we define  $B_{\sigma,j}^\chi$  to be the matrix  $B_{\sigma,j}$  but in which the column corresponding to integrals of the differential  $\psi_\chi = \frac{dz}{y_\chi}$  (with  $k = 0$ ) is replaced by  $\frac{u_{\chi,\sigma}}{o(\sigma)}$  times the integrals of  $\frac{\psi_\chi}{z - \lambda_{\sigma,j}}$ . Then the derivative  $\frac{\partial(\ln \det C)}{\partial \lambda_{\sigma,j}}$  can be evaluated as the sum  $\sum_{\{\chi \in \hat{A} | t_\chi \geq 2\}} \det B_{\sigma,j}^\chi / \det B_{\sigma,j}$ .*

*Proof.* As the Binomial Theorem allows us write any power  $(z - \lambda_{\sigma,j})^k$  appearing in  $B_{\sigma,j}$  as a linear combination of powers of  $z$  between 0 and  $k$ , the matrix  $B_{\sigma,j}$  is obtained from  $C$  via right multiplication by a matrix of determinant 1 (indeed, if the columns are ordered such that the differentials associated to a given character  $\chi$  are gathered together and appear in increasing order of  $k$ , this

matrix is upper triangular with 1s on the diagonal). Therefore  $\det B_{\sigma,j} = \det C$ , and it suffices to consider the derivative of  $\ln \det B_{\sigma,j}$  instead. The non-vanishing of that determinant reduces us to verifying that the derivative of  $\det B$  itself (without the  $\ln$ ) is  $\sum_{\chi} \det B_{\sigma,j}^{\chi}$ .

But if  $H$  is a square matrix of functions of a variable  $x$  then  $\frac{d(\det H)}{dx}$  equals  $\sum_l \det H_l$ , where  $H_l$  is the matrix  $H$  in which the  $l$ th column is replaced by its derivative. Indeed, expand  $H(x+h)$  as  $H(x) + hH'(x) + O(h^2)$ , write the determinant using the sum over permutations of products of entries of  $H$ , and consider the part that is linear in  $h$ . Hence for differentiating  $\det B_{\sigma,j}$  we shall need the determinants of the matrices obtained by replacing the differentials  $(z - \lambda_{\sigma,j})^k \psi_{\chi}$ , representing the columns of  $B_{\sigma,j}$ , by their derivatives with respect to  $\lambda_{\sigma,j}$ . The resulting matrix will be denoted  $B_{\sigma,j}^{\chi,k}$ . In the part where  $(z - \lambda_{\sigma,j})^k$  is differentiated (so that  $k > 0$ ), we just get  $k$  times another differential in the list. Hence this part does not contribute to  $\det B_{\sigma,j}^{\chi,k}$ . As for  $y_{\chi}$ , we recall from the proof of Corollary 4.4 and the paragraph preceding it that when  $P$  is neither a branching value nor a pole of  $z$ ,  $y_{\chi}(P)$  can be written as a function of  $P$  that does not depend on  $\lambda_{\sigma,j}$  times  $(z(P) - \lambda_{\sigma,j})^{u_{\chi,\sigma}/o(\sigma)}$ . It follows that the derivative of  $\frac{1}{y_{\chi}(P)}$  with respect to  $\lambda_{\sigma,j}$  yields just  $\frac{u_{\chi,\sigma}/o(\sigma)}{y_{\chi}(P)(z(P) - \lambda_{\sigma,j})}$ . For  $k > 0$  this part of the derivative again gives a multiple of another column, so that  $\det B_{\sigma,j}^{\chi,k} = 0$  for such  $k$ , and we know that only characters  $\chi$  with  $t_{\chi} \geq 2$  appear as columns by Proposition 1.2. As the matrix  $B_{\sigma,j}^{\chi,0}$  is just the required matrix  $B_{\sigma,j}^{\chi}$  by definition, this proves the lemma.  $\square$

The relation between the derivative from Lemma 6.1 and an expression from Corollary 5.5 is given in the following proposition.

**Proposition 6.2.** *For every non-trivial  $\sigma \in A$  and  $1 \leq j \leq r_{\sigma}$  we have*

$$\sum_{\{\chi \in \hat{A}, \chi(\sigma) \neq 1\}} \frac{f_{\chi,2}^{\chi}(\lambda_{\sigma,j})}{(f_{\chi,0}^{\chi})'(\lambda_{\sigma,j})} = -\frac{\partial \ln \det C}{\partial \lambda_{\sigma,j}}.$$

*Proof.* Lemma 4.7 states that  $\int_{P \in a_i} \omega_{\eta}(P, Q) = 0$  for every  $1 \leq i \leq g$  and every  $Q$ . In particular this holds for  $Q$  in the neighborhood of a branch point, for which we can expand  $\omega_{\eta}$  as in the first assertion of Lemma 5.2. Since the expansion is in the coordinate of  $Q$  and the integrals are with respect to  $P$ , the coefficient of every power of  $t$  must vanish after the integration. We restrict attention to powers of  $t$  that do not exceed  $o(\sigma) - 2$ , and we may assume that  $\eta(\sigma) \neq 1$  (for otherwise no such powers of  $t$  appear in the expansion from Lemma 5.2) and ignore the external multiplying coefficient. The resulting equality arises from the coefficient of  $t^{o(\sigma)-1-u_{\eta,\sigma}} dt$  (which is the only power having a non-zero coefficient in that lemma except for the error term), and takes the form

$$\frac{u_{\eta,\sigma}}{o(\sigma)} (f_{\eta,0}^{\eta})'(\lambda_{\sigma,j}) \int_{a_i} \frac{dz}{y_{\eta}(z - \lambda_{\sigma,j})} + \sum_{\chi \in \hat{A}} \sum_{l=0}^{t_{\chi}} f_{\chi,l}^{\eta}(\lambda_{\sigma,j}) \int_{a_i} \frac{(z - \lambda_{\sigma,j})^{l-2} dz}{y_{\chi}} = 0$$

for every  $1 \leq i \leq g$ , where the first integrand is  $\frac{\psi_\eta}{z-\lambda_{\sigma,j}}$  and the other ones are  $(z-\lambda_{\sigma,j})^{l-2}\psi_\chi$ .

We interpret these equations we get after dividing by the non-zero number  $(f_{\eta,0}^\eta)'(\lambda_{\sigma,j})$  as a set of  $g$  linear equations (one for every  $1 \leq i \leq g$ ) in which the indeterminates are the  $g$  expressions  $f_{\chi,l}^\eta(\lambda_{\sigma,j})/(f_{\eta,0}^\eta)'(\lambda_{\sigma,j})$ , expressions that we consider as the coordinates a vector  $\vec{x}$ . The matrix of coefficients is just the matrix  $B_{\sigma,j}$  from Lemma 6.1, so that this linear system is  $B_{\sigma,j}\vec{x} = \vec{y}$  where the  $i$ th coordinate of  $\vec{y}$  is  $-\frac{u_{\eta,\sigma}}{o(\sigma)} \int_{a_i} \frac{\psi_\eta}{z-\lambda_{\sigma,j}}$ . As  $B_{\sigma,j}$  is invertible, one can apply Cramer's rule, and see that  $f_{\chi,l}^\eta(\lambda_{\sigma,j})/(f_{\eta,0}^\eta)'(\lambda_{\sigma,j})$  can be written as  $-\det B_{\sigma,j}^{\eta,\chi,l} / \det B_{\sigma,j}$ , where  $B_{\sigma,j}^{\eta,\chi,l}$  is the matrix in which the column associated with the differential  $(z-\lambda_{\sigma,j})^{l-2}\psi_\chi$  is replaced by  $\frac{u_{\eta,\sigma}}{o(\sigma)}$  times the integrals of  $\frac{\psi_\eta}{z-\lambda_{\sigma,j}}$ . But for  $\chi = \eta$  and  $l = 2$  the matrix  $B_{\sigma,j}^{\eta,\chi,l}$  is just the matrix denoted  $B_{\sigma,j}^\eta$  in Lemma 6.1, which identifies the quotient associated with  $\eta$  in that lemma with minus the term associated with  $\eta$  in the left hand side of the required relation. Replacing the notation  $\eta$  by  $\chi$  also here, the restriction on  $\chi$  in Lemma 6.1 was  $t_\chi \geq 2$ , while here we demand that  $\chi(\sigma) \neq 1$ . But we recall from Corollary 4.4 that the polynomial  $f_{\chi,l}^\chi$  does not vanish identically only for  $l \leq t_\chi$ , so that adding the restriction  $t_\chi \geq 2$  does not affect any term with  $f_{\chi,2}^\chi$ . On the other hand, the matrix  $B_{\sigma,j}^\eta$  from Lemma 6.1 has a column multiplied by  $\frac{u_{\chi,\sigma}}{o(\sigma)}$ , so that if  $\chi(\sigma) = 1$  its determinant vanishes. Therefore the sum both here and in that lemma considers only characters  $\chi$  satisfying both  $t_\chi \geq 2$  and  $\chi(\sigma) \neq 1$ , so that they indeed coincide. This completes the proof of the proposition.  $\square$

At this point we shall invoke a variational formula due to Rauch, for stating which we shall need some notation. Assume that  $X$  is a compact Riemann surface, which lies in a family of compact Riemann surfaces parametrized holomorphically by the complex number  $h$  in a small disc, such that the genus  $g$  of the surface  $X_h$  is independent of  $h$ , and that we have chosen paths  $a_i$  and  $b_i$  on the real topological space underlying all the  $X_h$ s (and  $X = X_0$ ). Assume further that there is a map  $z$  defined from each  $X_h$  to  $\mathbb{CP}^1$ , and such that as a continuous map from the common underlying real surface of the  $X_h$ s to  $\mathbb{CP}^1$ ,  $z$  does not vary with  $h$ . Choose, at every point  $P \in X$ , a coordinate  $t_P$  such that  $t_P^{e_P}$  is a function of  $z$ , where  $e_P$  is the ramification index of  $z$  at  $P$ . By letting  $h$  vary we may write the function  $t_P^{e_P}$  of  $z$  on  $X_h$  as  $\sum_{\nu=0}^{e_P} c_{P,\nu}(h)t_{P,h}^\nu$ , where  $t_{P,h}$  is a coordinate around  $P$  on  $X_h$  depending holomorphically on  $h$ . The functions  $c_{P,\nu}$  of  $h$  satisfy the conditions  $c_{P,e_P}(h) = 1$  for every  $h$  while  $c_{P,\nu}(0) = 0$  for every  $0 \leq \nu < e_P$ . By taking the power  $\frac{1}{e_P}$  and expanding, we obtain that  $t_P$  is  $t_{P,h} [1 + h \sum_{\nu=0}^{e_P-1} c'_{P,\nu}(0)/t_{P,h}^{e_P-\nu}] + O(h^2)$ . The period matrix  $\tau(h)$  of  $X_h$  is a holomorphic function of  $h$ , the derivative of whose entries at 0 we seek, and  $\{v_s\}_{s=1}^g$  is the basis for the differentials of the first kind of  $X$  satisfying the usual conditions with respect to integration along the  $a_i$ s. Then the *Rauch variational formula*, as appearing in Equation (29) of [Ra], is the following.

**Theorem 6.3.** *Under the conditions and notations stated above we have the equality*

$$\left. \frac{d\tau_{rs}(h)}{dh} \right|_{h=0} = \sum_{P \in X} \frac{2\pi i}{e_P} \sum_{\nu=0}^{e_P-2} \frac{c'_{P,e_P-2-\nu}(0)}{\nu!} \frac{d^\nu}{dt^\nu} \left( \frac{v_r}{dt_P} \cdot \frac{v_s}{dt_P} \right) \Big|_{t=0}.$$

Note that the sum over  $P$  is finite, since the sum over  $\nu$  is trivial unless  $e_P \geq 2$ , i.e., unless  $P$  is a branch point of  $z$  (for  $X$ ). Moreover, the formula is invariant (as it should be) under replacing  $t_{P,h}$  by a translate  $t_{P,h} + s(h)$  with  $s(0) = 0$ : Indeed, such a variation of  $t_{P,h}$  will change  $c_{P,\nu}$  by a multiple  $s(h)^{e_P-\nu}$ , so that  $c_{P,e_P}$  remains unaffected, the derivative of  $c'_{P,e_P-1}(0)$  gathers a multiple of  $s'(0)$ , and while for the other functions  $c_{P,\nu}$  with  $\nu < e_P - 1$ , differing by  $O(h^{e_P-\nu})$ , their derivatives at  $h = 0$  (which are those appearing in Theorem 6.3) will not be affected.

Using Proposition 6.2 and Theorem 6.3, Corollary 5.5 takes the following form.

**Corollary 6.4.** *As a function of  $\lambda_{\sigma,j}$  the derivative of the logarithm of theta constant  $\theta[e](0, \tau)$ , namely  $\frac{\partial \ln \theta[e](0, \tau)}{\partial \lambda_{\sigma,j}}$ , equals*

$$\frac{1}{2} \frac{\partial \ln \det C}{\partial \lambda_{\sigma,j}} + \frac{n}{2} \sum_{(\rho,i) \neq (\sigma,j)} \frac{2\phi_{h+d\mathbb{Z}}(s) + \phi_{h+d\mathbb{Z}}(0) + \frac{(o(\sigma)-1)(o(\rho)-1)}{4}}{o(\sigma)o(\rho)(\lambda_{\sigma,j} - \lambda_{\rho,i})}.$$

*Proof.* We begin by showing that the sum over  $Q \in f^{-1}(\lambda_{\sigma,j})$  in Corollary 5.5, multiplied by the external coefficient there, is the derivative  $\frac{\partial \tau_{rs}}{\partial \lambda_{\sigma,j}}$  divided by  $2\pi i$ . Indeed, let us substitute the perturbation of  $\tau_{rs}$  that is associated to  $\lambda_{\sigma,j}$  in Theorem 6.3. As we have seen, only branch points have to be considered, and in our case the coordinate  $t$  around a branch point lying over  $\lambda_{\rho,i}$  satisfies  $t^{o(\rho)} = z - \lambda_{\rho,i}$ . Assuming first that  $(\rho, i) \neq (\sigma, j)$ , we observe that a small perturbation in  $\lambda_{\sigma,j}$  does not affect neither the location of the branch point in question nor the equality relating its local coordinate to  $z$ . Therefore for every such point  $P$  we have  $c_{Q,\nu} = \delta_{\nu,o(\rho)}$  (a constant), so that all the derivatives at 0 vanish and there is no contribution to  $\frac{\partial \tau_{rs}}{\partial \lambda_{\sigma,j}}$  from any of these points. Taking now a point  $Q$  over  $\lambda_{\sigma,j}$ , we find that for a small  $h$  the coordinate  $t_{Q,h}$  satisfies  $t_{Q,h}^{o(\sigma)} = z - \lambda_{\sigma,j} - h$ , so that our original coordinate  $t_Q$  satisfies  $t_Q^{o(\sigma)} = t_{Q,h}^{o(\sigma)} + h$ . Once again we have  $c_{Q,\nu} = \delta_{\nu,o(\rho)}$  for  $\nu > 0$  (and vanishing derivatives), but  $c_{Q,0}(h) = h$  has the derivative 1 at  $h = 0$  (and throughout). Substituting this into the formula from Theorem 6.3, observing that the external coefficient from Corollary 5.5 is precisely the quotient of  $\frac{1}{e_Q}$  and  $\frac{1}{\nu!}$  from that theorem (indeed,  $e_Q = o(\sigma)$  and  $\nu = o(\sigma) - 2$  for getting the index 0 of the non-vanishing  $c$ -derivative), we find that  $\frac{\partial \tau_{rs}}{\partial \lambda_{\sigma,j}}$  is indeed  $2\pi i$  times the asserted expression. This proves the claim.

Next, we observe that  $\frac{\partial^2 \ln \theta[e]}{\partial z_r \partial z_s}$  decomposes as  $\frac{\partial^2 \theta[e]}{\partial z_r \partial z_s} / \theta[e] - \frac{\partial \theta[e]}{\partial z_r} \cdot \frac{\partial \theta[e]}{\partial z_s} / \theta[e]^2$ . But for any value of  $e$  appearing in Corollary 5.5, all of the first derivatives of  $\theta[e]$  at  $z = 0$  vanish by Proposition 3.5, so that only the first term survives. We now recall that any theta function is known to satisfy the *heat equation*, stating that its image under  $\frac{\partial^2}{\partial z_r \partial z_s}$  coincides with its image under  $\frac{\partial}{\partial \tau_{rs}}$ , multiplied by  $4\pi i$  if  $r = s$  and by  $2\pi i$  otherwise. The fact that the terms are symmetric with respect to interchanging  $r$  and  $s$ , we may assume  $r \leq s$  and have the same multiplier  $4\pi i$  in both cases  $r = s$  and  $r < s$ . Observing that we divide by  $\theta[e]$  itself and that the two instances of  $2\pi i$  cancel, we find that the right hand side in Corollary 5.5 reduces to  $2 \sum_{1 \leq r \leq s \leq g} \frac{\partial \ln \theta[e](0, \tau)}{\partial \tau_{rs}} \cdot \frac{\partial \tau_{rs}}{\partial \lambda_{\sigma, j}}$ , which becomes, when we consider the theta constant  $\theta[e](0, \tau)$  as a function of  $\lambda_{\sigma, j}$  via the dependence of  $\tau$  on that parameter, just twice the required derivative. By comparing it with half the right hand side of Corollary 5.5, in which the term with the polynomials  $f_{\chi, 2}^X$  is expressed as in Proposition 6.2, we obtain the desired equality. This proves the corollary.  $\square$

For obtaining well-defined functions on the moduli space, independent of the choice of orderings on the branching values, we want the exponents appearing in the final expressions to be integral and even. For this we briefly investigate the integrality properties of the functions  $\phi_{h+d\mathbb{Z}}$  from Equation (5).

**Lemma 6.5.** *If  $d$  is co-prime to 6 then  $\phi_{h+d\mathbb{Z}}(s) \in \mathbb{Z}$  for every  $s \in \mathbb{Z}$ . In case  $d$  is odd but divisible by 3 we have  $\phi_{h+d\mathbb{Z}}(s) \in \frac{h}{3} + \mathbb{Z}$  (which is not integral since 3 does not divide  $h$ ). For even  $d$  not divisible by 3 the number  $\phi_{h+d\mathbb{Z}}(s)$  lies in  $\frac{1+2s}{4} + \mathbb{Z}$ . Finally, if  $d$  is divisible by 6 then  $\phi_{h+d\mathbb{Z}}(s)$  belongs to  $\frac{1+2s}{4} - \frac{h}{3} + \mathbb{Z}$ .*

*Proof.* Forgetting about the group  $A$ , we find that  $\phi_{h+d\mathbb{Z}}(s)$  would be obtained also from a cyclic group of order  $d$  generated by an element  $\sigma$ , provided that  $\rho$  is taken to be  $\sigma^h$ , if the divisor we take to represent  $e$  does not include  $\lambda_{\sigma, j}$  but in which  $\lambda_{\rho, i}$  appears to the power  $s$ . Indeed, in this case it would be  $d$  times the resulting  $q_e$  parameter. Moreover, the characters of this cyclic group are generated by the one sending  $\sigma$  to  $\mathbf{e}(\frac{1}{d})$ , and if  $\chi$  is the  $u$ th power of this character (in which we take  $0 \leq u < d$ ) then  $u_{\chi, \sigma} = u$ . Since  $u_{\chi, \rho}$  would then be  $d$  times the fractional part of  $\frac{hu}{d}$ , we can write  $\phi_{h+d\mathbb{Z}}(s)$  as  $d \sum_{u=0}^{d-1} (\frac{u}{d} - \frac{d-1}{2d}) (\{\frac{hu+s}{d}\} - \frac{d-1}{2d})$ . The similarity to Dedekind sums is now obvious, so that for the integrality we adapt the argument from Section 3 of [Rd]. First, the sum of the right multipliers alone was seen to vanish (see the proofs of Propositions 3.5 and 5.3 or Lemma 3.6), and by separating the fractional part the expression for  $\phi_{h+d\mathbb{Z}}(s)$  becomes  $\sum_{u=0}^{d-1} u (\frac{hu}{d} - \lfloor \frac{hu+s}{d} \rfloor + \frac{2s-(d-1)}{2d})$ . The integral parts contribute an integer, the term with  $h$  gives  $\frac{h(d-1)(2d-1)}{6}$ , the one with  $s$  becomes  $\frac{(d-1)(2s-d+1)}{4}$ , and the sum is  $\frac{(d-1)[6s-3d+3+4dh-2h]}{12}$ . If  $d$  is odd then both multipliers in the numerators are even, so that up to integers we get  $\frac{d-1}{2} \cdot \frac{h(2d-1)}{3}$ . For  $d$  not divisible by 3 one of the multipliers is divisible by 3, and otherwise the residue of  $\frac{h(d-1)(2d-1)}{2}$  modulo 3 is easily seen to be  $-h$ . Now, for even  $d = 2k$  it suffices (again, up to integers) to subtract  $\frac{6s-6k+3+8kh-2h}{12}$  from

$\frac{k(6+4kh-4h)}{12}$ , yielding just  $\frac{2h+4k^2h-3-6s}{12}$  plus another integer. Now, if 3 does not divide  $d$  (or equivalently  $k$ ) then  $\frac{4k^2h}{12} \in \frac{4h}{12} + \mathbb{Z}$ , and as  $h$  is odd for even  $d$  the resulting expression  $\frac{6h-3-6s}{12}$  is indeed just  $\frac{1+2s}{4}$  plus an integer. Otherwise we have to evaluate  $\frac{2h-3-6s}{12}$  up to integers, by adding to which the integer  $\frac{6(1-h)+12s}{12}$  (recall again that  $h$  is odd) we obtain  $\frac{3+6s-4h}{12}$  which is the required value. This proves the lemma.  $\square$

Recall now that  $n = |A|$  and that  $m$  denotes the exponent of  $A$ , and that they have the same prime divisors.

**Theorem 6.6.** *Let  $\Delta$  be a divisor given in the form of Equation (1), with  $p = 1$  and such that the equalities from Theorem 1.4 hold (so that  $\Delta$  is of degree  $g - 1$  and  $r(-\Delta) = 0$ ), and set  $e = u(\Delta) + K$ . Then there exists a complex number  $\alpha_e$ , which is independent of the branching values  $\lambda_{\sigma,j}$ , such that  $\theta[e]^{8m}(0, \tau)$  equals*

$$\alpha_e (\det C)^{4m} \prod_{(\sigma,j) < (\rho,i)} (\lambda_{\sigma,j} - \lambda_{\rho,i})^{\frac{4mn}{o(\sigma)o(\rho)}} \left[ 2\phi_{h+d\mathbb{Z}}(s) + \phi_{h+d\mathbb{Z}}(0) + \frac{(o(\sigma)-1)(o(\rho)-1)}{4} \right]$$

for every choice of the branching values. Here the  $<$  sign denotes an arbitrary full order on the set of pairs  $(\sigma \in A, 1 \leq j \leq r_\sigma)$ , and once the indices  $\sigma, \rho, j$ , and  $i$  are given,  $d$  is defined to be  $|\langle \sigma \rangle \cap \langle \rho \rangle|$ ,  $h$  is determined by the equality  $\rho^{o(\rho)/d} = (\sigma^{o(\sigma)/d})^h$ ,  $s$  is  $\beta_{\rho,i} - h\beta_{\sigma,j}$  modulo  $d\mathbb{Z}$ , and  $\phi_{h+d\mathbb{Z}}$  is the function defined in Equation (5).

The arbitrary order appears in Theorem 6.6 in order to verify that each couple of pairs is counted exactly once. The assumption that  $e$  is defined independently of the branch points, which is used implicitly in that theorem, should be understood as follows. By Proposition 2.5, for a fixed choice of branching values, the value of  $e$  in  $J(X)$  is torsion of some explicit order dividing  $2n$ . Writing  $J(X)$  as the quotient of  $\mathbb{C}^g$  modulo the lattice generated by the columns of  $I$  and  $\Pi$ , we find that  $e$  has some entries in  $(\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^{2g}$  in the resulting coordinates. These entries are the ones that we consider as independent of the choices of the branching values.

*Proof.* Corollary 6.4 implies that the function  $\ln \theta[e](0, \tau)$  has the same derivative as

$$\frac{\ln \det C}{2} + \frac{n}{2} \sum_{(\rho,i) \neq (\sigma,j)} \frac{2\phi_{h+d\mathbb{Z}}(s) + \phi_{h+d\mathbb{Z}}(0) + \frac{(o(\sigma)-1)(o(\rho)-1)}{4}}{o(\sigma)o(\rho)} \ln(\lambda_{\sigma,j} - \lambda_{\rho,i})$$

with respect to the variable  $\lambda_{\sigma,j}$  (where  $e$  is considered as fixed as described above, which must be the case since  $e$  can be viewed as a continuous map of  $\lambda_{\sigma,j}$  into the discrete set of torsion points of order dividing  $2n$  in  $J(X)$ ). Recall that the orders  $o(\sigma)$  and  $o(\rho)$  must divide  $m$  and  $n$ , and we claim that multiplying the combination in brackets by 8 always produces an even integer. Indeed, the expression not involving  $\phi_{h+d\mathbb{Z}}$  and the parts from  $\mathbb{Z}$  and  $\frac{1}{4}\mathbb{Z}$  in

Lemma 6.5 become even, and so is the total contribution  $\frac{-16h}{3} + \frac{-8h}{3}$  in case  $3|d$ . In total,  $8m \ln \theta[e](0, \tau)$  and  $8m$  times the latter expression differ by an additive number that is independent of  $\lambda_{\sigma,j}$ , so that after exponentiation we obtain that  $\theta[e]^{8m}(0, \tau)$  itself is some number  $\beta_{e,\sigma,j}$ , which may depend on all the other branching values (and  $e$ ) but not on  $\lambda_{\sigma,j}$ , times  $(\det C)^{4m}$  times the asserted product over  $(\rho, i) \neq (\sigma, j)$ . We may write  $\beta_{e,\sigma,j}$  as another number  $\alpha_{e,\sigma,j}$  times any function  $f$  of the other branching values  $\lambda_{\rho,i}$  of our choice, as long as  $f$  never vanishes. We do this by taking  $f$  to be the product over all the pairs appearing in the desired expression, in which none of the pairs of indices coincides with the  $\sigma$  and  $j$  with which we work. This gives us the desired equality, but with  $\alpha_{e,\sigma,j}$  instead of  $\alpha_e$ . However, as the left hand side and the product on the right hand side (and  $(\det C)^{4m}$ ) are all independent of the choice of  $\sigma$  and  $j$ , we find that  $\alpha_{e,\sigma,j}$  is the same number for all  $\sigma$  and  $j$ , so that we may denote it simply by  $\alpha_e$ . As  $\alpha_e$  does not depend on the value of  $\lambda_{\sigma,j}$  for any  $\sigma$  and  $j$  (since it equals  $\alpha_{e,\sigma,j}$ ), this completes the proof of the theorem.  $\square$

Note that in general the expression in brackets may have a true denominator of 4, in case both  $o(\sigma)$  and  $o(\rho)$  are even but  $d$  is odd. In the other cases with either  $o(\sigma)$  or  $o(\rho)$  (or both) even, this number may be either integral or half-integral (depending on the residues modulo 4), but when both these orders are odd it is always integral. However, we can strengthen Theorem 6.6 by applying finer observations as to obtain the minimal number by which we have to multiply the differential equation from Corollary 6.4 for making all the multiplying coefficients even integers. The result is as follows. Set the number  $\varepsilon$  to be the minimal positive integer such that  $2^\varepsilon \text{lcm}\{m, \frac{n}{m}\}/m$  is divisible by 8, while for odd  $n$  we set  $\varepsilon_2 = 2$ . Then Theorem 6.6 also holds if we replace the exponent of the theta constant by  $2^\varepsilon m \text{lcm}\{m, n/m\}/n$ , that of  $\det C$  by half that number, and the coefficient preceding the brackets by  $\frac{2^\varepsilon m \text{lcm}\{m, n/m\}}{o(\sigma)o(\rho)}$ . We also note that it is difficult to say explicit things about the value of  $\alpha_e$ , since  $\det C$  depends on the normalization of the functions  $y_\chi$  (which may be replaced by their scalar multiples).

## 7 Well-Definition on the Moduli Space

Let us now investigate what can be said about the dependence of  $\alpha_e$  on the characteristic  $e$ . In the non-singular  $Z_n$  case considered in [Na] it is shown in that reference that the  $n$ th power of this coefficient gives the same value for every  $e$ . The more elementary references [EiF] (non-singular case,  $n = 2$  and  $n = 3$ ), [EbF] (non-singular case, any  $n$  as in [Na]) and [FZ] have shown such an independence of  $e$  for several families of non-singular  $Z_n$  curves, while in [Z] the general case of fully ramified  $Z_n$  curves is established, depending on a conjecture about the operation of certain operators being transitive.

Let us begin our investigation in this direction using a method that is orthogonal to that of these references.

**Proposition 7.1.** *Let  $e$  and  $\epsilon$  be characteristics, obtained via  $u + K$  from the divisors  $\Delta$  and  $\Xi$ , both of which are given in terms of Equation (1) and satisfy the conditions of Theorem 1.4, and assume that the coefficients in that equation are  $\beta_{\sigma,j}$  for  $\Delta$  and  $\kappa_{\sigma,j}$  for  $\Xi$ . Assume that for each  $\sigma \in A$  and  $0 \leq i < o(\sigma)$ , the sets  $\{j | \beta_{\sigma,j} = i\}$  and  $\{j | \kappa_{\sigma,j} = i\}$  have the same cardinality. Then  $\alpha_e = \alpha_\epsilon$ .*

*Proof.* The fact that the right hand side in Theorem 6.6 contains only even powers of differences implies that it is a well-defined function on the moduli space of  $A$ -covers of  $\mathbb{CP}^1$  with fixed  $r_\sigma$  for every non-trivial  $\sigma \in A$ . This also holds for even powers of  $\det C$ , since  $z$  and the differentials  $\psi_\chi$  are also well-defined only in terms of the sets  $\{\lambda_{\sigma,j} | 1 \leq j \leq r_\sigma\}$ . Therefore when we take a closed path in that space, the value at the initial point of the path coincides with that of the end point. Moreover, since all the symmetric groups are generated by permutations, it suffices to prove the result in case  $\Xi$  is obtained from  $\Delta$  by interchanging two values  $\lambda_{\sigma,j}$  and  $\lambda_{\sigma,k}$  (i.e., if  $\kappa_{\sigma,j} = \beta_{\sigma,k}$ ,  $\kappa_{\sigma,k} = \beta_{\sigma,j}$ , and  $\kappa_{\rho,i} = \beta_{\rho,i}$  wherever  $\rho \neq \sigma$  or  $\rho = \sigma$  and  $i \notin \{j, k\}$ ).

Assuming that  $\Xi$  and  $\Delta$  differ by such an interchange, we consider  $\alpha_e$  as the quotient having  $\theta^{8m}[u(\sum_\rho \sum_{i=1}^{r_\rho} \beta_{\rho,i} \lambda_{\rho,i} - f^{-1}(\infty)) + K](0, \tau)$  (which is  $\theta^{8m}[u(\Delta) + K](0, \tau) = \theta^{8m}[e](0, \tau)$ ) in the numerator and  $(\det C)^{4m}$  times the polynomial from Theorem 6.6 as the denominator. Take a point  $\mu \in \mathbb{C}$  that is not a branching value, and consider a smooth path from  $\lambda_{\sigma,j}$  to  $\mu$  not passing through any other branch point. Theorem 6.6 shows that if we move  $\lambda_{\sigma,j}$  along this path, while fixing all the other points  $\lambda_{\rho,i}$ , the value of  $\alpha_e$  will remain unaffected. After reaching  $\mu$ , we now fix  $\mu$  and the points  $\lambda_{\rho,i}$  except for  $\lambda_{\sigma,k}$ , and we move the latter point along a path to the initial value of  $\lambda_{\sigma,j}$  (again, not passing through any other branching value or  $\mu$ ). After having done so, we move again the branch point now having value  $\mu$  via a path (not going via any other branching value once more) ending in the initial value of  $\lambda_{\sigma,k}$ . Theorem 6.6 implies that the last two steps also leave  $\alpha_e$  invariant. But checking the value of the quotient after carrying out this operation, we get the same point on the moduli space (the set of  $r_\rho$  branching values attached to  $\rho$  remained the same, also for  $\rho = \sigma$ ), the numerator became  $\theta^{8m}[u(\Xi) + K](0, \tau) = \theta^{8m}[\epsilon](0, \tau)$ , and the denominator is again the product of  $(\det C)^{4m}$  and the polynomial associated with  $\Xi$  (or with  $\epsilon$ ). As this value is therefore  $\alpha_\epsilon$ , we obtain the desired equality. This completes the proof of the proposition.  $\square$

In fact, Proposition 7.1 suffices for proving the full Thomae formula for the case considered in [Na] (without the additional computations), since it was seen in that reference (as well as in Section 2.2 of [FZ]) that all the relevant divisors can be obtained from one another by permuting the branching values. As another consequence, recall that the moduli space for our type of coverings of  $\mathbb{CP}^1$  can be seen (after fixing the discrete parameters  $r_\sigma$  with  $\sigma \in A$ , hence also  $g$  by Proposition 1.1) can be described, when the map  $z$  is seen as fixed, in terms of the  $n - 1$  sets  $\{\lambda_{\sigma,j} | 1 \leq j \leq r_\sigma\}$  with non-trivial  $\sigma \in A$ . However, our description of  $e$  and  $\Delta$  depends on knowing which point of the  $\lambda_{\sigma,j}$  is associated



with each index  $j$ . The fact that  $\alpha_e$  is invariant under permuting the elements of each set  $\{\lambda_{\sigma,j} | 1 \leq j \leq r_\sigma\}$ , provided by Proposition 7.1, ensures that these constants are indeed well-defined on our moduli space. The extension to the remaining closed set, as well as the invariance under projective transformations of the map  $z$ , is done as follows.

**Proposition 7.2.** *Theorem 6.6 holds also when one of the branch points equals  $\infty$ . Moreover, letting a Möbius transformation act on all the branch points leaves the constant  $\alpha_e$  invariant.*

*Proof.* We have to determine the growth of both  $(\det C)^{4m}$  and the polynomial from Theorem 6.6 as  $\lambda_{\sigma,j} \rightarrow \infty$ . The notation will be simpler if we recall  $\lambda_{\sigma,j} - \lambda_{\rho,i}$  appears with the exponent  $4m(2q_e(\sigma, j; \rho, i) + n\gamma_{\sigma,\rho})$ , and we might ignore the multiplier  $4m$  in both of them. Now, recalling that  $y_\chi$  looks like  $(z - \lambda_{\sigma,j})^{u_{\chi,\sigma}/o(\sigma)}$  times expressions not depending on  $\lambda_{\sigma,j}$ , we find that the integrand  $z^l \psi_\chi$  with  $0 \leq l \leq t_\chi - 2$  grows as  $\lambda_{\sigma,j}^{-u_{\chi,\sigma}/o(\sigma)}$  when this parameter tends to  $\infty$  (and the integration path remains fixed, of course). As there are  $t_\chi - 1$  values of  $l$  for such  $\chi$ , we find that the order of  $\det C$  with respect to this growth of  $\lambda_{\sigma,j}$  is  $-\sum_{\chi \in \hat{A}} (t_\chi - 1) \frac{u_{\chi,\sigma}}{o(\sigma)}$  (the artificial inclusion of the character  $\chi = \mathbf{1}$  does not affect the value, since  $u_{\mathbf{1},\sigma} = 0$ ), which after substituting the value of  $t_\chi$  and the already known value of  $\sum_{\chi \in \hat{A}} \frac{u_{\chi,\sigma}}{o(\sigma)}$  becomes just  $\frac{n(o(\sigma)-1)}{2o(\sigma)} - \sum_{\rho \in A} nr_\rho \gamma_{\sigma,\rho}$ . As for the polynomial, by adding and subtracting the exponent that would correspond to the pair  $(\rho, i) = (\sigma, j)$  we obtain  $\sum_{\rho,i} (2q_e(\sigma, j; \rho, i) + n\gamma_{\sigma,\rho}) - 2q_e(\sigma, j; \sigma, j) + n\gamma_{\sigma,\sigma,j}$ . But the latter two numbers we evaluated as  $\frac{n(o(\sigma)-1)(o(\sigma)+1)}{6o(\sigma)^2}$  and  $\frac{n(o(\sigma)-1)(2o(\sigma)-1)}{6o(\sigma)^2}$  in the proofs of Lemmas 5.1 and 5.2, so that their sum  $\frac{n(o(\sigma)-1)}{2o(\sigma)}$  cancels the positive part of the order of  $\det C$ , while the sum of  $n\gamma_{\sigma,\rho}$ , which is independent of  $i$ , cancels with the negative part there. As for the remaining term, when we restrict our attention to a single summand  $q_\Delta(\sigma, j; \rho, i)$  in the definition of  $q_e(\sigma, j; \rho, i)$ , the proof of Proposition 3.2 shows that the sum over  $\rho$  and  $i$  is (up to a multiple depending on  $\sigma$  and  $j$ ) the expression  $\frac{\deg \Delta + n}{n} - \frac{g+n-1}{n}$  appearing there, which is known to vanish. Therefore the total denominator is bounded as  $\lambda_{\sigma,j} \rightarrow \infty$ , yielding in the limit the polynomial from which the terms involving  $\lambda_{\sigma,j}$  are omitted and a normalization of  $y_\chi$  and  $z^l \psi_\chi$  that produces the corresponding expressions on the  $A$ -cover of  $\mathbb{CP}^1$  in which  $\infty$  is branched and associated to  $\sigma$ .

As for the Möbius action, we recall that the theta constants are independent of  $z$  (they are just defined on  $X$  with respect to the action of  $A$ ), so that we only have to check  $\det C$  and the polynomial. It is well-known that the group  $PSL_2(\mathbb{C})$  of Möbius transformations is generated by translations, complex dilations, and the inversion, so that we check each of these separately. They polynomial is clearly invariant under translations, and as the proof of Lemma 6.1 allows us to replace  $z^l$  by  $(z - \mu)^l$  for any  $\mu$  without altering  $\det C$ , the invariance of the latter also follows since moving the paths  $a_i$  via replacing  $z$  by  $z + \mu$  back again is a homotopy which does not change the values of integrals of holomorphic differentials. As for dilations, the polynomial is homogenous, and its degree is

$2m[\sum_{\sigma,j}\sum_{\rho,i}(2q_e(\sigma,j;\rho,i)+n\gamma_{\sigma,\rho})-\sum_{\sigma,j}(2q_e(\sigma,j;\sigma,j)+n\gamma_{\sigma,\sigma})]$ , where the sum with fixed  $\sigma$  and  $j$  was seen above to give  $\sum_{\chi\in\hat{A}}(t_\chi-1)\frac{u_{\chi,\sigma}}{o(\sigma)}$ . Summing over  $\sigma$  and  $j$  and putting the multiplier back in we obtain just  $2m\sum_{\chi\in\hat{A}}t_\chi(t_\chi-1)$ . On the other hand, when multiplying every  $\lambda_{\sigma,j}$  by some non-zero constant  $c$ , we may move the paths  $a_i$  homotopically onto the path corresponding to  $cz$ , and then the variable change from  $z$  to  $cz$  in the new integral of  $z^l\psi_\chi$  would give us the initial integral, but divided by  $c$  raised to the power  $l+1-\sum_{\sigma,j}\frac{u_{\chi,\sigma}}{o(\sigma)}$ , where the latter sum is just  $t_\chi$ . This happens for every  $\chi$  and  $l$ , and as  $4m$  times the sum over  $0\leq l\leq t_\chi-1$  gives just  $2mt_\chi(t_\chi+1)-4mt_\chi$ , this power of  $c$  indeed cancels with the one from the homogeneity of the polynomial. Finally, for the inversion, assuming that none of the  $\lambda_{\sigma,j}$ s vanishes, we can write  $\frac{1}{\lambda_{\sigma,j}}-\frac{1}{\lambda_{\rho,i}}$  as  $\frac{\lambda_{\rho,i}-\lambda_{\sigma,j}}{\lambda_{\sigma,j}\lambda_{\rho,i}}$ , and gathering the powers of each  $\lambda_{\sigma,j}$  as above we find (using the fact that all the exponents are even) that doing so divides our polynomial by  $\prod_{\sigma,j}\lambda_{\sigma,j}^{4m\sum_{\chi\in\hat{A}}(t_\chi-1)u_{\chi,\sigma}/o(\sigma)}$ . As for the integrals, up to a modification of the paths so that they do not pass through zeros of  $z$  as well, we may replace  $z$  by  $\frac{1}{z}$ , so that  $z^l dz$  becomes  $-\frac{dz}{z^{l+2}}$  and the modified expression  $\prod_{\sigma,j}(z-\frac{1}{\lambda_{\sigma,j}})^{u_{\chi,\sigma}/o(\sigma)}$  becomes, up to some root of unity, the original one divided by  $\prod_{\sigma,j}(z\lambda_{\sigma,j})^{u_{\chi,\sigma}/o(\sigma)}$ . As the total exponent of  $z$  is  $t_\chi$ , we get the differential  $z^{t_\chi-2-l}\psi_\chi$  (which is one of our basis elements from Proposition 1.2 since  $0\leq l\leq t_\chi-2$ ), multiplied by  $\prod_{\sigma,j}\lambda_{\sigma,j}^{u_{\chi,\sigma}/o(\sigma)}$ . For the determinant we take again the sum over  $\chi$  and  $l$  of that expression, producing a total product that cancels with the one from modifying the polynomial as we have already seen. This completes the proof of the proposition.  $\square$

The actual meaning of Proposition 7.2 is that the constants  $\alpha_e$  are well-defined on the moduli space (the full one, without the restriction about  $\infty$ ) of  $A$ -covers of  $\mathbb{CP}^1$  with parameters  $r_\sigma$  for non-trivial  $\sigma\in A$ , considered as Riemann surfaces, not including the data of the map  $z$ . The first part of Proposition 7.2 could have also been proved using with the fact that when we take the limit  $\lambda_{\sigma,j}\rightarrow\infty$  in Corollaries 5.5 and 6.4 the terms involving that parameter simply disappear, but the evaluations were seen to be useful for proving the second part as well.

Next, [Z] shows that for general fully ramified  $Z_n$  curves, under a certain transitivity conjecture, the quotients between the  $4n^2$ th powers of theta constants and appropriate polynomials are independent of the choice of characteristics. To see the relation between the expressions from [Z] and our Theorem 6.6, a certain condition has to be verified.

**Lemma 7.3.** *Consider the argument  $s$  of the function  $\phi_{h+d\mathbb{Z}}$  in Equation (5) as a number between 0 and  $d-1$ . Then adding  $h$  to the argument coincides with adding  $s-\frac{d-1}{2}$  to the value, i.e., the equality  $\phi_{h+d\mathbb{Z}}(s+h)=\phi_{h+d\mathbb{Z}}(s)+s-\frac{d-1}{2}$  holds for every such  $d$ ,  $h$ , and  $s$ .*

*Proof.* In the difference between  $\phi_{h+d\mathbb{Z}}(s+h)$  and  $\phi_{h+d\mathbb{Z}}(s)$  in Equation (5), we can cancel  $1-e(kh/d)$ , and get just  $\sum_{0\neq k\in\mathbb{Z}/d\mathbb{Z}}\frac{-e(ks/d)}{1-e(-k/d)}$ . But the proof

of Proposition 5.3 shows that for each  $k$  this quotient is the difference between  $\sum_{v=1}^{d-1} \frac{v}{d} \mathbf{e}(-kv/d)$  and  $\sum_{v=d-s}^{d-1} \mathbf{e}(-kv/d)$ . Interchanging the summation order back, we find that since  $\mathbf{e}(-v/d)$  is a true root of unity for every  $v$  (since we exclude  $v = 0$ ), the sum  $\sum_{0 \neq k \in \mathbb{Z}/d\mathbb{Z}} \mathbf{e}(-kv/d)$  equals  $-1$  for every such  $v$ . The difference in question therefore reduces to  $\sum_{v=d-s}^{d-1} 1 - \sum_{v=1}^{d-1} \frac{v}{d}$ , which easily gives the required value. This proves the lemma.  $\square$

It follows from Lemma 7.3 that  $-2\phi_{h+d\mathbb{Z}}$  satisfies the defining condition for the function denoted  $f_h^{(d)}$  (with our indices) in [Z], and as the latter function is normalized to vanish at 0, we can write  $f_h^{(d)}(s) = 2\phi_{h+d\mathbb{Z}}(0) - 2\phi_{h+d\mathbb{Z}}(s)$ . Moreover, altering the exponent of  $\lambda_{\sigma,j} - \lambda_{\rho,i}$  by a constant depending only on  $\sigma$  and  $\rho$  (but not  $i$  and  $j$ ) does not affect the invariance from [Z]. Now, in [Z] the equality involves  $\theta^{2en^2}[u(\Delta) + K](0, \tau)$  and  $\lambda_{\sigma,j} - \lambda_{\rho,i}$  raised to a power which in our notation equals  $en(c(\sigma, \rho) - f_h^{(d)}(s))$  for some number  $e$  depending on the parity of  $n$ , while in our Theorem 6.6 the power of the theta constant is  $8m$  and the coefficient multiplying  $\phi_{h+d\mathbb{Z}}(s)$  is  $\frac{8mn}{o(\sigma)o(\rho)}$ . Recalling that in the fully ramified  $Z_n$  case we have  $m = n$  and  $o(\sigma)$  equals  $n$  as well for every  $\sigma$  with  $r_\sigma > 0$ , the ratio between these multipliers reduces to  $n$ . Comparing that with the ratio  $-2n$  associated with  $f_h^{(d)}(s)$ , the relation between the latter function and ours show that up to raising the formulae from both references by some finite simple powers, the polynomials obtained here are the same as those from [Z]. We also remark that  $\phi_{h+d\mathbb{Z}}(s)$  is invariant under replacing  $h$  by its inverse modulo  $d$  and multiplying  $s$  by minus that inverse (as is seen by a change of the summation index  $k$ ), so that this number is indeed an invariant of the unordered pair  $(\sigma, j; \rho, i)$  (in correspondence with Lemma 4.4 of [Z]).

Moreover, setting  $s = 0$  in the formula for  $\phi_{h+d\mathbb{Z}}$  in Equation (5) produces  $d \cdot s(h, d) + \frac{d-1}{4}$ , where  $s(h, d)$  is the classical Dedekind sum (this can be seen either from Equation (5) itself with the alternative formula for  $s(h, d)$ , or using the expressions from the proof of Lemma 6.5, but noting that we also include the summand with  $u = 0$  and the we subtract  $\frac{d-1}{2d}$  rather than  $\frac{1}{2}$ ). The recursive formula for Dedekind sums (see, e.g., Equation (3) of [Rd]) translates to  $\phi_{h+d\mathbb{Z}}(0)$  being equal  $\frac{d^2+h^2+3hd-3d-3h+1}{12h} - \frac{d}{h}\phi_{d+h\mathbb{Z}}(0)$  (when  $h$  is considered as lying between 0 and  $d$ ). Combining this with the recursive formula for  $f_h^{(d)}(s)$  appearing in Theorem 6.4 of [Z], we find that

$$\phi_{h+d\mathbb{Z}}(s) = \frac{d^2 + h^2 + 3hd - 3d - 3h + 1 - 6s(d + h - 1 - s)}{12h} - \frac{d}{h}\phi_{d+h\mathbb{Z}}(s)$$

(provided that  $0 \leq s < d$  as well), and then in the rightmost summand we may replace both  $d$  and  $s$  by their residues modulo  $h$ . This gives a recursive argument for evaluating  $\phi_{h+d\mathbb{Z}}(s)$  for every  $d, h$ , and  $s$ . For example, it follows that  $\phi_{1+d\mathbb{Z}}(s) = \frac{d^2-1-6s(d-s)}{12}$  (and in particular  $\phi_{1+d\mathbb{Z}}(0) = \frac{d^2-1}{12}$ ) for every  $d \geq 1$ , since  $\phi_{0+1\mathbb{Z}}(s) = 0$  for every  $s$ . Another feature of  $\phi_{h+d\mathbb{Z}}(s)$  in comparison to the appropriate generalization of  $s(h, d)$  is, apart from the description of the

denominators in Lemma 6.5, is the equality  $\sum_{s \in \mathbb{Z}/d\mathbb{Z}} \phi_{h+d\mathbb{Z}}(s) = 0$  holding for every  $d$  and  $h$ .

Proposition 7.1 and the preceding paragraph form enough evidence for us to pose the following conjecture.

**Conjecture 7.4.** *There is some integral power  $N$ , depending only on  $n$  and  $m$ , such that the  $N$ th powers of the coefficients  $\alpha_e$  from Theorem 6.6 all give the same constant.*

If Conjecture 7.4 holds, then the appropriate power of the quotient is a global constant characterizing the moduli space in question, which depends only on  $A$  and the numbers  $r_\sigma$  for  $\sigma \in A$  (this information already includes the genus  $g$  by Proposition 1.1). We remark that as the tools applied in both [Na] and [Z] (as well as the references of the latter) use in a central manner the possibility to add a branch point to an invariant divisor and obtain (perhaps up to linear equivalence) another invariant divisor. Since such an operation can be carried out only for fully ramified  $Z_n$  curves, tools for proving the invariance required for Conjecture 7.4 are yet to be discovered, and the proof of this conjecture will be left for future investigation.

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